Cohomology and Poincaré duality

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Contents

Chapter 1

Cohomology and Universal Coefficient Theorem

1.1 Course description

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Reference books:

- Allen Hatcher, "Algebraic Topology", CUP 2002.
- Greenberg and Harper, "Algebraic Topology: a first course" : Addison-Wesley 1981
- William Fulton, "Algebraic Topology: a first course" : Springer 1995, GTM 153
- Bott and Tu, "Differential forms in Algebraic Topology": Springer-Verlag 1999, GTM 82

Prerequisites: MA3H6 Algebraic Topology, MA3H5 Manifolds

1.2 Cohomology

Recall homology group is defined in the following lines. First, one forms a *chain complex*, which is a sequence of abelian groups and homomorphisms:

$$C_\bullet : \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

The choices of the chain complex could be singular, simplicial, or cellular. Homomorphisms ∂_n are called *boundary map* with the property $\partial_n \circ \partial_{n+1} = 0$ for all n.

The homology groups (of the chain complex) are defined as

$$H_n(C_\bullet) = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)},$$

where the elements of $Z_n(C_\bullet) := \ker \partial_n$ are called n-cycles, and elements of $B_n(C_\bullet) := \mathrm{Im}\partial_{n+1}$ are called n-boundaries.

The cohomology groups are defined in the similar lines as a dual object of homology groups. We first define the *cochain group*

$$C^n = Hom(C_n, G) = C_n^*$$

as the dual of the chain group C_n. And then define the *coboundary map* $\delta_n : C^n \to C^{n+1}$ as the dual of boundary map: For $f \in C^n$, define $(\delta_n f)(x) := f(\partial_{n+1} x)$ for $x \in C_{n+1}$. It is easy to check $\delta_n \circ \delta_{n-1} = 0$.

Then cohomology groups are define as

$$H^n(C^\bullet) = \frac{Z^n(C^\bullet)}{B^n(C^\bullet)},$$

where the elements of $Z^n(C^\bullet) := \ker \delta_n$ are called n-cocycles, and elements of $B^n(C^\bullet) := \mathrm{Im}\delta_{n-1}$ are called n-coboundaries.

$$C^\bullet : \cdots \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} \cdots$$

Let us calculate several examples. Take the coefficient group $G = \mathbb{Z}$.

Example 1.2.1. *Let $C_\bullet$ be the chain complex $\cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots$ where $\mathbb{Z}$ is at k^{th} position. Then the homology groups are*

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

The corresponding cochain complex $C^\bullet$ is $\cdots \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \cdots$ where $\mathbb{Z}$ is still at k^{th} position. Then

$$H^n(C^\bullet) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

Example 1.2.2. *Let $C_\bullet$ be the chain complex $\cdots \to 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0 \to \cdots$ where $\mathbb{Z}$ are at k^{th} and $(k-1)^{th}$ positions. Then the homology groups are*

$$H_n(C_\bullet) = \begin{cases} \mathbb{Z}_m & n = k-1 \\ 0 & n \neq k-1 \end{cases}$$

The corresponding cochain complex $C^\bullet$ *is* $\cdots \leftarrow 0 \leftarrow \mathbb{Z} \overset{m}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow \cdots$ *where* $\mathbb{Z}$ *are still at* k^{th} *and* $(k-1)^{th}$ *position. But*

$$H^n(C^\bullet) = \begin{cases} \mathbb{Z}_m & n = k \\ 0 & n \neq k \end{cases}$$

These examples show the difference of the free part $\mathbb{Z}$ and torsion part $\mathbb{Z}_n$. Actually if C_n is free abelian for all n and $H_n(C)$ is finitely generated for all n. And $H_n(C) = F_n \oplus T_n$ where F_n is free abelian and T_n is torsion. Then $H^n(C) = F_n \oplus T_{n-1}$. This is the simplest form of Universal coefficient theorem which determines cohomology groups with arbitrary coefficients from homology with $\mathbb{Z}$ coefficients.

Remark 1.2.3. *For non-abelian group* G*, we could still define (co)homology, but the point is that usually* $H^n(C;G)$ *do not have a group structure when* $n > 0$*, since* $Im\delta$ *need not to be a normal subgroup of* $ker\delta$*.*

1.3 Universal Coefficient Theorem

1.3.1 Chain map and chain homotopy

Definition 1.3.1. *A map* $f : C \to D$ *of chain complexes* C *and* D *is a sequence of homomorphisms* $f = \{f_n : C_n \to D_n\}$ *such that* $\partial_n \circ f_n = f_{n-1} \circ \partial_n$*.*

Proposition 1.3.2. *The chain map* $f : C \to D$ *induces a homomorphism* $f_* : H_n(C) \to H_n(D)$ *of homology groups.*

Proof. By definition, it f_n takes cycles in $Z_n(C)$ to cycles in $Z_n(D)$ and takes boundaries to boundaries. Hence it induces a homomorphism $f_* : H_n(C) \to H_n(D)$. □

Definition 1.3.3. *Two maps of chain complexes* $f, g : C \to D$ *are chain homotopic (denoted by* $f \simeq g$*) if there exists a sequence of maps* $T = \{T_n : C_n \to D_{n+1}\}$ *such that*

$$\partial_{n+1} \circ T_n + T_{n-1} \circ \partial_n = f_n - g_n.$$

We will save the subscripts for boundary maps later on.

Proposition 1.3.4. *If* f *and* g *are chain homotopic, then their induced homomorphisms of homology are equal.*

Proof. Let T be a chain homotopy. For any $z_n \in Z_n(C)$, we have

$$f_*[z_n] - g_*[z_n] = [g_n(z_n) - f_n(z_n)] = [\partial \circ T(z_n) + T \circ \partial(z_n)] = [\partial T(z_n)] = 0.$$

□

So chain homotopy is an equivalence relation.

Definition 1.3.5. *Two chain complexes C and D are called chain homotopy equivalent ($C \simeq D$), if there are chain maps $f : C \to D$ and $g : D \to C$ such that*

$$g \circ f \simeq id_C : C \to C, f \circ g \simeq id_D : D \to D.$$

Each of them is called a chain homotopy equivalence.

The next result follows from Propostion 1.3.4.

Proposition 1.3.6. *Chain homotopy equivalence induces isomorphism of homology groups. So if $C \simeq D$, then $H_*(C) = H_*(D)$.*

The reverse is also true if C and D are complexes of abelian groups.

All the above discussion works for cochain complexes with apparent modifications.

1.3.2 Hom functor

Notice that the functor $Hom(\cdot, G)$ is the key for cohomology. By definition, $Hom(H, G)$ is the set of all homomorphisms from H to G. It is an abelian group as well, and called *homomorphism group*. Let us look at it in a more abstract viewpoint. It is a countravariant functor, which means $f : A \to B$ induces $f^* : Hom(B, G) \to Hom(A, G)$ and if furthermore we have $g : B \to C$ then $(g \circ f)^* = f^* \circ g^*$. The above discussion tells us $Hom(\cdot, G)$ is also a countravariant functor from Chain complexes to cochain complexes.

It has the following properties:

- Splitting exactness: $Hom(A_1 \oplus A_2, G) = Hom(A_1, G) \oplus Hom(A_2, G)$;
- Moreover for direct sum: $Hom(\oplus_i A_i, G) = \Pi_i Hom(A_i, G)$;
- left exactness: If $A \stackrel{f}{\to} B \stackrel{g}{\to} C \to 0$ is an exact sequence, then the induced sequence

  $$Hom(A, G) \stackrel{f^*}{\longleftarrow} Hom(B, G) \stackrel{g^*}{\longleftarrow} Hom(C, G) \longleftarrow 0$$

  is exact.

Proofs are left as exercises. The third item is the key for later applications.

Example 1.3.7. $Hom(\mathbb{Z}_m, G) = \{g \in G | mg = 0\}$. *In fact, take Hom for exact sequence $0 \to \mathbb{Z} \stackrel{m}{\to} \mathbb{Z} \to \mathbb{Z}_m \to 0$, we obtain*

$$0 \leftarrow G \stackrel{m}{\leftarrow} G \leftarrow Hom(\mathbb{Z}_m, G) \leftarrow 0.$$

By item three, the sequence is exact except possibly for the leftmost. So $Hom(\mathbb{Z}_m, G) = \ker(G \stackrel{m}{\to} G)\{g \in G | mg = 0\}$. And when $mG \neq G$, the sequence is not exact. In general, Hom functor does NOT keep exactness.

To state UCT, we need to introduce a functor $Ext(H, G)$, which measures the failure of Hom to be an exact functor. It is defined from a free resolution of the abelian group H: $0 \to F_1 \to F_0 \to H \to 0$, with $F_i = 0$ for $i > 1$.This could be obtained in the following way. Choose a set of generators for H and let F_0 be a free abelian group with basis in one-to-one correspondence with these generators. Then we have a surjective homomorphism $f_0 : F_0 \to H$. The kernel of f_0 is free as a subgroup of a free abelian group. We let F_1 be the kernel and the inclusion to F_0 as f_1. It is an exact chain complex sequence. In summary, constructing a free resolution is equivalent to choosing a presentation for A.

Take its dual cochain complex by $Hom(F, G)$, we may lose its exactness, so could have its cohomology group, temporarily denoted by $H^n(F; G)$. For the above constructed resolution, $H^n(F; G) = 0$ for $n > 1$. So the only interesting group is $H^1(F; G)$. As we will show, it is independent of the resolution. There is a standard notation for that: $Ext(H, G)$. The element in this group could also be interpreted as the isomorphism class of extensions of G by H, i.e. $0 \to G \to J \to H \to 0$.

Finally, the Universal coefficient theorem is ready to state.

Theorem 1.3.8. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then for each n, there is a natural short exact sequence:*

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{h} Hom(H_n(C), G) \to 0.$$

The sequence splits, so that

$$H^n(C; G) = Ext(H_{n-1}(C), G) \oplus Hom(H_n(C), G).$$

But the splitting is not natural.

1.3.3 Proof of Universal coefficient theorem, step 1

There is a natural choice of free resolution of the homology group

$$0 \to B_n(C) \xrightarrow{i_n} Z_n(C) \xrightarrow{q} H_n(C) \to 0.$$

So the $H^1(F; G)$ for this free resolution is exactly $coker(i_n^* : Hom(Z_n, G) \to Hom(B_n, G))$ by definition. Now, let us prove Universal coefficient theorem in two steps.

Step 1: derive the split short exact sequence

$$0 \to Coker i_{n-1}^* \to H^n(C; G) \xrightarrow{h} Hom(H_n(C), G) \to 0.$$

Step 2: Prove $H^1(F; G)$ only depends on H and G, but not the resolution. So $coker i_{n-1}^* = Ext(H_{n-1}, G)$.

We start with Step 1.

Lemma 1.3.9. *There is a natural homomorphism*

$$h : H^n(C; G) \to Hom(H_n(C), G).$$

Proof. We first have the map in the cycle level:

For any cocycle $\alpha \in Z^n$ and any cycle $z \in Z_n$, we let $h(\alpha)(z) = \alpha(z)$. $\alpha \in Z_n$ means $\delta\alpha = 0$, i.e $\alpha\partial = 0$. In other words, α vanishes on B_n. So h descends to a map from Z^n to $Hom(H_n, G)$.

Next if $\alpha \in B^n$, then $\alpha = \delta\beta = \beta\partial$. Hence α is zero on Z_n. Thus there is a well defined quotient map $h([\alpha])([z]) = \alpha(z)$ from $H^n(C; G)$ to $Hom(H_n(C), G)$.

This is a homomorphism since:

$$h([\alpha + \beta])([z]) = (\alpha + \beta)(z) = \alpha(z) + \beta(z) = h[\alpha]([z]) + h[\beta]([z]).$$

□

Now there is a split short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

It splits since B_{n-1} is free (for any generator of B_{n-1}, one could map it to a preimage of ∂. So it is not canonically chosen.). Thus we have $p : C_n \to Z_n$ whose restriction to Z_n is identity.

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & B_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 \end{array}$$

Since the dual of a split short exact sequence is a split short exact sequence (the splitting exactness of Hom), the following commutative diagram has exact rows,

$$\begin{array}{ccccccccc} 0 & \longleftarrow & Z_n^* & \longleftarrow & C_n^* & \xleftarrow[\delta]{} & B_{n-1}^* & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longleftarrow & Z_{n+1}^* & \longleftarrow & C_{n+1}^* & \xleftarrow[\delta]{} & B_n^* & \longleftarrow & 0 \end{array}$$

This is a part of a short exact sequence of chain complexes. From this, we have a long exact sequence (because the differential of the complexes B and Z are trivial)

$$B_n^* \xleftarrow{i_n^*} Z_n^* \longleftarrow H^n(C; G) \longleftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^*.$$

The connection homomorphism is i_n^* because of definition: one takes an element of Z_n^*, pulls back to C_n^*, applies δ to get an element in C_{n+1}^*, then pull back to B_n^*. So we first extend a homomorphism $f : Z_n \to G$ to $f' : C_n \to G$, then composes it with ∂, finally view it as a map from B_n. So it is nothing but the restriction of f from Z_n to B_n. We could also see from its dual operation: given $b \in B_n$, so $b = \partial c$, then the first step maps it to c, second takes ∂, thus get b back which is in Z_n. The composition is the inclusion i_n.

Hence we have

$$0 \leftarrow Keri_n^* \leftarrow H^n(C;G) \leftarrow Cokeri_{n-1}^* \leftarrow 0.$$

The last step for Step 1 is

Lemma 1.3.10. $\ker(i_n^*) = Hom(H_n(C), G)$

Proof. Because the elements of $\ker(i_n^*)$ are homomorphisms $Z_n \to G$ that vanish on B_n, the same as homomorphisms $H_n = Z_n/B_n \to G$. □

Under this identification, the natural map h is the map $0 \leftarrow Keri_n^* \leftarrow H^n(C;G)$. And the short exact sequence splits because of the induced map p^*.

1.4 Ext functor

Recall that to finish Step 2, we only need to prove that for any two (2-step) free resolutions of abelian group H, the homology groups are isomorphic. So the notation $Ext(H, G)$ is well defined.

We have the following

Lemma 1.4.1. *Given free resolutions F and F' of abelian groups H and H', then every homomorphism α could be extended to a chain map from F to F':*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\cdots & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0
\end{array}$$

Furthermore, any two such chain maps extending α are chain homotopic.

Proof. Since F_i's are free, it suffices to define α_i on a basis of F_i. Given $x \in F_0$, then $\alpha(f_0(x)) \in H'$. Since f_0' is surjective, we have x' such that $f_0'(x') = \alpha(f_0(x))$. We define $\alpha_0(x) = x'$.

To define α_1, for $x \in F_1$, $\alpha_0(f_1(x))$ lies in $Imf_1' = Kerf_0'$ since $f_0'\alpha_0 f_1 = \alpha f_0 f_1 = 0$. So define $\alpha_1(x) = x'$ such that $\alpha_0(f_1(x)) = f_1'(x')$. Other α_i could be constructed inductively by a similar way.

To check any such chain maps are chain homotopic, we will only give a proof for the case of 2-step free resolutions, i.e when $F_i = F'_i = 0$ for $n > 1$. This is the case we need. The general case is similar and see page 194 of Hatcher for details.

If β_i is another extension of α, then we want to find $T_0 : F_0 \to F'_1$ and $T_{-1} : H \to F'_0$ such that $\alpha_i - \beta_i = f'_{i+1}T_i + T_{i-1}f_i$ for $i = 0, 1$. We let $T_{-1} = 0$. We let $T_0(x) = x'$ such that $f'_1(x') = \alpha_0(x) - \beta_0(x)$. This could be done because $f'_0\beta_0(x) = f'_0\alpha_0(x) = \alpha f_0(x)$ and $Imf'_1 = kerf'_0$. Hence $\alpha_0 - \beta_0 = f'_1T_0$. To check $\alpha_1 - \beta_1 = T_0 f_1$, we only need to check the relation after composing f'_1 since it is injective. It is nothing but $f'_1(\alpha_1 - \beta_1) = (\alpha_0 - \beta_0)f_1$. □

Corollary 1.4.2. *For any two free resolutions F and F' of H, $H^n(F; G) = H^n(F'; G)$.*

Proof. It follows from above lemma and (cohomology version of) Proposition 1.3.6 by taking $\alpha = id : H \to H$ and by looking at the composition of two chain maps, one from F to F' and the other from F' to F. □

Hence we finished step 2 and thus the proof of Theorem 1.3.8.

Now the calculation of cohomology groups is reduced to that of Ext. So next will be the properties of it.

Proposition 1.4.3. *Computation properties:*

1. $Ext(H \oplus H', G) = Ext(H, G) \oplus Ext(H', G)$
2. $Ext(H, G) = 0$ *if H is free abelian*
3. $Ext(\mathbb{Z}_n, G) = G/nG$.

Proof. For the first, we take direct sum of the free resolution.

For the second, we use $0 \to H \to H \to 0$ as the resolution to calculate.

For the last, we use

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0.$$

and the calculation in Example 1.3.7. □

Corollary 1.4.4. *If the homology groups of chain complex C of free abelian groups are finitely generated and $H_n(C) = F_n \oplus T_n$ where F_n is free and T_n is torsion, then $H^n(C; \mathbb{Z}) = F_n \oplus T_{n-1}$.*

Proof. It follows from Theorem 1.3.8 and Proposition 1.4.3, and the fact $Hom(\mathbb{Z}_m, \mathbb{Z}) = 0$, $Hom(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. □

Next property shows how Ext fucntor remedies the left exactness of Hom functor.

Proposition 1.4.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Then there is a six-term exact sequence $0 \to Hom(C, G) \to Hom(B, G) \to Hom(A, G) \to Ext(C, G) \to Ext(B, G) \to Ext(A, G) \to 0$.*

1.4.1 Universal coefficient theorem and Künneth formula for homology

Instead of Hom, we apply $\otimes$ to a free resolution $0 \to F_1 \to F_0 \to H \to 0$. This operation is right exact, so by similar idea as Ext, we use Tor to measure its non-exactness, i.e. Tor is the first (and the only non-trivial) homology of the new complex. Then we have

Theorem 1.4.6. *If C is a chain complex of free complex of free abelian groups, then there are natural short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to Tor(H_{n-1}(C), G) \to 0$$

for all n. These sequences split, though not naturally.

The proof is almost identical to that of Theorem 1.3.8 after establishing corresponding properties for Tor. See Hatcher's Appendix A to Chapter 3 for details.

Finally, we want to remark that the universal coefficient theorem in homology is a special case of the Künneth theorem. We first introduce teh tensor products of chain complexes.

Let (C, ∂) and (D, ∂) be chain complexes, where C_i and D_i are zero for $i < 0$. The *tensor product* of chain complexes is

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q,$$

with differential

$$\partial(c_p \otimes d_q) = (\partial c_p) \otimes d_q + (-1)^p c_p \otimes (\partial d_q).$$

It indeed defines a chain complex since

$$\partial\partial(c_p \otimes d_q) = \partial((\partial c_p) \otimes d_q + (-1)^p c_p \otimes (\partial d_q)) = (-1)^{p-1} \partial c_p \otimes \partial d_q + (-1)^p \partial c_p \otimes \partial d_q = 0.$$

Tensor product of chain maps is defined as $(f \otimes g)(c_p \otimes d_q) = (f_p c_p) \otimes (g_q d_q)$. Please check it commutes with ∂. We also know that chain homotopy is kept by tensor product.

Theorem 1.4.7 (Künneth formula)**.** *For a free chain complex C and an arbitrary chain complex D, there is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor(H_p(C), H_q(D)) \to 0.$$

It splits but not canonically.

To get Theorem 1.4.6, we take D as the complex $D_0 = G$ and $D_i = 0$ for $i \neq 0$.

1.5 Singular cohomolgy

This is an explicit example of choosing the chain complex and cochain complex.

Recall that a standard singular n-simplex is the convex set $\Delta^n \subset \mathbb{R}^{n+1}$ consisting of all $(n+1)$-tuples of real numbers $(v_0, \cdots, v_n)$ with $v_i \geq 0, v_0 + \cdots + v_n = 1$. A singular n-simplex in X is a continuous map $\sigma : \Delta^n \to X$. The singular chain group $S_n(X)$ is the free abelian group generated by the singular n-simplices. The boundary homomorphism $\partial : S_n(X) \to S_{n-1}(X)$ is defined as $\partial(\sigma) = \sum_i (-1)^i \sigma|[v_0, \cdots, \hat{v_i}, \cdots, v_n]$. Then we could define singular homology and denoted by $H_n(X)$.

Fix coefficient group G, we set $S^n(X) = Hom(S_n(X), G)$ be the cochain group. The coboundary map $\delta : S^n(X; G) \to S^{n+1}(X; G)$ is the dual of ∂. So any $\phi \in S^n(X; G)$ $\delta\phi$ is the composition $S_{n+1}(X) \xrightarrow{\partial} S_n(X) \xrightarrow{\phi} G$. Explicitly, for $\sigma : \Delta^{n+1} \to X$,

$$\delta\phi(\sigma) = \sum_i (-1)^i \phi(\sigma|[v_0, \cdots, \hat{v_i}, \cdots, v_{n+1}]).$$

For a pair (X, A), we could also talk about relative groups. Let $S_n(X, A) = \frac{S_n(X)}{S_n(A)}$ and $S^n(X, A; G) = Hom(S_n(X, A), G)$. The elements of $S^n(X, A; G)$ are n-cochains taking the value 0 on singular n-simplices in A. Hence we have relative groups $H^*(X, A; G) := H^*(S^*(X, A; G))$. A map of pairs $f : (X, A) \to (Y, B)$ induces homomorphisms $f^* : S^n(Y, B; G) \to S^n(X, A; G)$ and $f^* : H^n(Y, B; G) \to H^n(X, A; G)$.

We could describe relative cohomology groups in terms of exact sequences. Recall that the short exact sequence

$$0 \to S_n(A) \xrightarrow{i} S_n(X) \xrightarrow{j} S_n(X, A) \to 0$$

splits. By splitting exactness of Hom functor, the dual

$$0 \leftarrow S^n(A; G) \xleftarrow{i^*} S^n(X; G) \xleftarrow{j^*} S^n(X, A; G) \leftarrow 0$$

is a split exact sequence as well. Since i^* and j^* commute with δ, so it induces a long exact sequence of cohomolgy groups

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \longrightarrow \cdots.$$

Let us describe the connecting homomorphism $H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G)$. For $\phi \in Z^n(A; G)$, we first extend it to a cochain $\bar{\phi} \in S^n(X; G)$, by taking value 0 on singular simplices not in A. Then $\delta^X(\bar{\phi}) = \bar{\phi}\partial \in S^{n+1}(X; G)$. It is in $S^{n+1}(X, A; G)$ because original ϕ is a cocycle in A, i.e taking value 0 in $B_n(A)$, which means $\delta^X(\bar{\phi}) = \bar{\phi}\partial$ takes value 0 on $S_{n+1}(A)$. Finally it is in $Z^{n+1}(X, A; G)$ because $\delta^{X,A}(\delta^X \bar{\phi}) = \delta^X(\delta^X \bar{\phi}) = 0$. Its class $[\delta^X \bar{\phi}] \in H^{n+1}(X, A; G)$ is $\delta[\phi]$.

A more general long exact sequence is for a triple (X, A, B), induced by

$$0 \leftarrow S^n(A, B; G) \xleftarrow{i^*} S^n(X, B; G) \xleftarrow{j^*} S^n(X, A; G) \leftarrow 0.$$

When B is a point, it induces the long exact sequence for reduced cohomology.

1.6 The Eilenberg-Steenrod Axioms for cohomology

For simplicity, let us omit the coefficient G in out notation.

A cohomology theory consists of 3 functions:

1. For any integer n and any pair of spaces (X, A), we have an abelian group $H^n(X, A)$.
2. For any integer n and any map of pairs $f : (X, A) \to (Y, B)$ (f maps A to B), we obtain homomorphism $f^* : H^n(Y, B) \to H^n(X, A)$.
3. For any integer n and any pair (X, A), we have a connecting homomorphism $\delta : H^n(A) \to H^{n+1}(X, A)$.

These functions satisfy the following 7 axioms:

1. Unit: If $id : (X, A) \to (X, A)$ is the identity, id^* is the identity.
2. Composition: $(g \circ f)^* = f^* \circ g^*$.
3. Naturality: Given $f : (X, A) \to (Y, B)$, the following diagram commutes:

$$\begin{array}{ccc} H^n(A) & \xleftarrow{f|_A^*} & H^n(B) \\ \downarrow \delta & & \downarrow \delta \\ H^{n+1}(X, A) & \xleftarrow{f^*} & H^{n+1}(Y, B) \end{array}$$

4. Exactness: The following sequence is exact:

$$\cdots \longrightarrow H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \longrightarrow \cdots.$$

   Here $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$ are inclusions.

5. Homotopy: If $f \simeq g$ are homotopic maps of pairs, then $f^* = g^*$.
6. Excision: Given (X, A) and $U \subset X$ such that $\bar{U} \subset int(A)$. Then the inclusion $i : (X \backslash U, A \backslash U) \to (X, A)$ induces isomorphisms in cohomology.

7. Dimension: Let P be a one-point space, then

$$H^n(P) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$$

There are many generalized cohomology theories which satisfy all the axioms except probably the dimension axiom. The Eilenberg-Steenrod uniqueness theorem says that there is a unique cohomology theory satisfying all the axioms in the category of finite CW complexes. Unfortunately the proof is beyond the scope of current course. But we could prove a weaker version.

Suppose $H^n(X, A)$ and $K^n(X, A)$ are cohomology theories, and $\phi : H^n(X, A) \to K^n(X, A)$ is a *natural transformation* of cohomology theories, i.e. it commutes with induced homomorphisms and with coboundary homomorphisms in long exact sequence of pairs.

Theorem 1.6.1 (Weak form of Eilenberg-Steenrod Uniqueness)**.** *Suppose $\phi : H^n(X) \to K^n(X)$ is an isomorphism when $X = \{pt\}$. Then ϕ is an isomorphism for any finite CW complex.*

We left the proof to the end of next section.

A few more words about Eilenberg-Steenrod axioms. If we want to prove uniqueness for a larger category, more conditions are needed for the uniqueness. For the category of CW complexes, an eighth axiom is added to guarantee its uniqueness. This is included in the discussion of Hatcher's book.

Milnor Addition: Let $X = \sqcup X_\alpha$ be disjoint union. Then the induced homomorphism $\prod_\alpha i_\alpha^* : H^n(X) \to \prod_\alpha H^n(X_\alpha)$ is an isomorphism.

This axiom has force only if there are infinitely many X_α. The finite sum case is a corollary of the following Mayer-Vietoris Sequence. There are also examples of cohomology theories which are not additive (by James and Whitehead).

1.7 Mayer-Vietoris Sequences

By using Eilenberg-Steenrod Axioms, we can form the Mayer-Vietoris Sequence. The model in our mind is $X = A \cup B$ with A and B open in X. But for the ease of application, we use the following setting.

Definition 1.7.1. *For $U \subset A \subset X$, the map $(X \backslash U, A \backslash U) \to (X, A)$ is an excision if the induced homomorhism $H^n(X, A) \to H^n(X \backslash U, A \backslash U)$ is an isomorphism for all n.*

Example 1.7.2. *The inclusion* $(D^+, S^{n-1}) \to (S^n, D^-)$ *is an excision. We cannot apply the Excision Axiom directly. But look at the exact sequences from exactness axiom corresponding to the two pairs, we have* $H^{i+1}(S^n, D^-) = H^{i+1}(S^n) = H^i(S^{n-1}) = H^{i+1}(D^+, S^{n-1})$ *when* $i > 0$ *by noticing* $H^n(D) = H^n(pt)$*. The rest of two identities also follow from the exact sequence by knowing* $H^0(D) = H^0(S^n) = G$*.*

Definition 1.7.3. *Suppose* $A, B \subset X$*, such that*

- $A \cup B = X$
- *Both* $(A, A \cap B) \to (X, B)$ *and* $(B, A \cap B) \to (X, A)$ *are excisions.*

Then $(X; A, B)$ *is an excisive triad.*

Example 1.7.4. *Let* $X = A \cup B$ *with* A *and* B *open in* X*. Then* $(A, A \cap B) = (X \backslash U, B \backslash U)$ *where* $U = B \backslash (A \cap B)$*. Since* $X \backslash U = A$*,* U *is closed and* $U \subset B = int(B)$*. Then the excision axiom shows this is an excision.*

Example 1.7.5. $(S^n; D^+, D^-)$ *is an excisive triad.*

Theorem 1.7.6. *Suppose* $(X; A, B)$ *is an excisive triad. Then there is a long exact sequence*

$$\cdots \longrightarrow H^n(X) \overset{(j_A^*, j_B^*)}{\longrightarrow} H^n(A) \oplus H^n(B) \overset{i_A^* - i_B^*}{\longrightarrow} H^n(A \cap B) \overset{\delta}{\longrightarrow} H^{n+1}(X) \to \cdots$$

Recall that a similar MV sequence is derived for homology as the long exact sequence associated to the short exact sequence

$$0 \to C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A + B) \to 0.$$

We could prove it similarly for singular cohomology say, but bot for a general cohomology theory.

We need the following algebraic lemma:

Lemma 1.7.7 (Barratt-Whitehead)**.** *Suppose we have the following commutative diagram with exact rows*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \alpha_{n-1}} & & \\ \cdots & \longrightarrow & A'_n & \xrightarrow{f'_n} & B'_n & \xrightarrow{g'_n} & C'_n & \xrightarrow{h'_n} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

in which every third map $\gamma_i : C_i \to C'_i$ *is an isomorphism. Then there is a long exact sequence*

$$\cdots \longrightarrow A_n \overset{(f_n, \alpha_n)}{\longrightarrow} B_n \oplus A'_n \overset{\beta_n - f'_n}{\longrightarrow} B'_n \overset{h_n \gamma_n^{-1} g'_n}{\longrightarrow} A_{n-1} \longrightarrow \cdots.$$

This is a standard diagram chasing argument. We would like to leave this as an exercise. So we only sketch the proof.

Proof. This is a chain complex is easy to check in terms of commutativity of the diagram. To check the exactness, we have 3 parts:

1. $ker(\beta_n - f'_n) \subset im(f_n, \alpha_n)$. Assume $\beta_n(b) = f'_n(d)$. By commutativity and exactness at B_n, we have $a \in A_n$ such that $f_n(a) = b$. We have $\alpha_n(a) - d \in ker f'_n = im h'_{n+1}$. So $h'_{n+1}(c') = \alpha_n(a) - d$. Let $a' = a - h_{n+1}\gamma_{n+1}^{-1}(c')$. Then $(f_n(a'), \alpha_n(a')) = (b, d)$.

2. $ker(h_n\gamma_n^{-1}g'_n) \subset im(\beta_n - f'_n)$. Let $e \in ker(h_n\gamma_n^{-1}g'_n)$. By exactness at C_n, we have $b \in B_n$ such that $g_n(b) = \gamma_n^{-1}g'_n(e)$. So $\beta_n(b) - e \in ker g'_n = im f'_n$. Choose any element d in it, we have $e = \beta_n(b) - f'_n(d)$.

3. $ker(f_{n-1}, \alpha_{n-1}) \subset im(h_n\gamma_n^{-1}g'_n)$. First find an element in C_n. Then an element in B'_n. □

Proof. (of the theorem) We have

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{n-1}(X) & \xrightarrow{j_B^*} & H^{n-1}(B) & \xrightarrow{\delta} & H^n(X,B) & \xrightarrow{h_n} & H^n(X) & \longrightarrow \cdots \\
& \downarrow j_A^* & & \downarrow i_B^* & & \downarrow \gamma_n & & \downarrow j_A^* & \\
\cdots \longrightarrow & H^{n-1}(A) & \xrightarrow{i_A^*} & H^{n-1}(A\cap B) & \xrightarrow{\delta} & H^n(A, A\cap B) & \xrightarrow{h'_n} & H^n(A) & \longrightarrow \cdots
\end{array}$$

γ_n is an isomorphism since excisive triad. Hence we have the MV sequence by Barratt-Whitehead. □

Example 1.7.8 (Cones and suspensions)**.** *Given a space X, the cone of X is*

$$C(X) := \frac{X \times [0,1]}{(x,1) \sim (y,1), \forall x, y \in X}$$

and suspension as

$$\Sigma(X) := \frac{X \times [0,1]}{(x,0) \sim (y,0), (x,1) \sim (y,1), \forall x, y \in X}.$$

Then $(\Sigma(X); C_+(X), C_-(X))$ is an excisive triad. Use MV sequence and $C_+(X) \simeq C_-(X) \simeq pt$, we have $H^k(X) = H^{k+1}(\Sigma X)$. If X is connected, $H^1(\Sigma X) = 0$ and $H^0(\Sigma X) = \mathbb{Z}$.

Especially $\Sigma S^n = S^{n+1}$, so it could be used to calculate the cohomology of S^n inductively. Complete the calculation.

Given a continuous map $f : S^{n-1} \to A$ for $n \geq 1$. We have

$$X = C(f) := A \cup_f D^n = \frac{A \sqcup D^n}{f(x) \sim x, \forall x \in S^{n-1}}$$

Notice that if $f \simeq g : S^{n-1} \to A$, then $C(f) \simeq C(g)$. Also f extends to a map $f : D^n \to C(f)$.

Proposition 1.7.9. *1. The inclusion $A \to X$ induces isomorphisms $H^q(A) = H^q(X)$ for $q \neq n, n-1$.*

2. There is an exact sequence

$$0 \to H^{n-1}(X) \to H^{n-1}(A) \xrightarrow{f^*} H^{n-1}(S^{n-1}) \to H^n(X) \to H^n(A) \to 0$$

Proof. Use sequence for pairs (or MV sequence). For both items, use $H^i(X, A) = H^i(D^n, S^{n-1}) = H^{i-1}(S^{n-1})$ and previous calculation for $H^*(S^n)$. □

Example 1.7.10. *Recall $\mathbb{C}P^n = \frac{\mathbb{C}^{n+1}\backslash 0}{\sim}$ where $x \sim y$ if there exists $\lambda \neq 0$ such that $\lambda x = y$.*

We write $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_f D^{2n}$, where $f : S^{2n-1} \to \mathbb{C}P^{n-1}$ is the natural quotient map. The above sequence tells us for $0 < m < 2n$,

$$0 \to H^{2n-1}(S^{2n-1}) \to H^{2n}(\mathbb{C}P^n) \to 0,$$

$$0 \to H^m(\mathbb{C}P^n) \to H^m(\mathbb{C}P^{n-1}) \to 0.$$

By induction and $H^0(\mathbb{C}P^n) = \mathbb{Z}$ since it is path connected, we have

$$H^m(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & m = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Now we can finish the proof of Theorem 1.6.1

Proof. We use induction. The dimension axiom gives us the dimension 0 case. Assume it is done for all complexes with dimension less that or equal to $n-1$, and X be a dimension n cell complex. So X is obtained by attaching n-cells to an $(n-1)$ cell complex. So we could do these attachment one by one (but A will possibly have dimension n then).

Hence the statement is reduced to the following: Suppose the result is true for A, prove the statement is true for $X = C(f) := A \cup_f D^n$. Proposition 1.7.9 item 1 tells us $\phi : H^q(X) \to K^q(X)$ is an isomorphism for $q \neq n-1, n$. For the rest, look at

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^{n-1}(X) & \longrightarrow & H^{n-1}(A) & \longrightarrow & H^{n-1}(S^{n-1}) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \simeq & & \downarrow \alpha & & \downarrow & & \downarrow \simeq & & \\ 0 & \longrightarrow & K^{n-1}(X) & \longrightarrow & K^{n-1}(A) & \longrightarrow & K^{n-1}(S^{n-1}) & \longrightarrow & K^n(X) & \longrightarrow & K^n(A) & \longrightarrow & 0 \end{array}$$

As we have shown in the suspension calculation, α is also an isomorphism. So five lemma completes the proof. □

The five lemma also shows us $H^*(X, A) = K^*(X, A)$.

Notice that for infinite CW complexes, we need Milnor additivity in the above argument. A telescope argument could reduce infinite dimensional case to finite dimensional case.

Chapter 2

Products

In this chapter, we take the coefficients in a commutative ring R with a unit.The most common choices are $\mathbb{Z}$, $\mathbb{Z}_m$ or $\mathbb{Q}$.

2.1 Cup Product for singular cohomology

There is a product structure in cohomolgy.We start with singular cohomology. Let $\sigma : \Delta^{m+n} \to X$ be a singular simplex. By the *front m-face* or ${}_m\sigma$, we mean $\sigma|[v_0, \cdots, v_m]$. Similarly, for the *back n-face* or σ_n, we mean $\sigma|[v_m, \cdots, v_{m+n}]$.

Then we define the cup product at chain level:

Definition 2.1.1. *Given $\phi \in S^m(X)$ and $\psi \in S^n(X)$, the cup product $\phi \cup \psi \in S^{m+n}(X)$ is defined by*

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \cdots, v_m]) \cdot \psi(\sigma|[v_m, \cdots, v_{m+n}]).$$

To get intuition for this definition, think about $S^1 \vee S^1 \vee S^2$ and T^2. Details in class.

Lemma 2.1.2. *For $\phi \in S^m(X)$ and $\psi \in S^n(X)$, we have*

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^m \phi \cup \delta\psi.$$

Proof. For $\sigma : \Delta^{m+n+1} \to X$, we have

$$\delta\phi \cup \psi(\sigma) = \sum_{i=0}^{m+1} (-1)^i \phi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{m+1}])\psi(\sigma|[v_{m+1}, \cdots, v_{m+n+1}])$$

$$(-1)^m(\phi \cup \delta\psi)(\sigma) = \sum_{i=m}^{m+n+1} (-1)^i \phi(\sigma|[v_0, \cdots, v_m])\psi(\sigma|[v_m, \cdots, \hat{v}_i, \cdots, v_{m+n+1}])$$

Add and cancel the last of the first sum and the first of the second sum, we have $\delta(\phi \cup \psi)(\sigma)$. □

Corollary 2.1.3. *There is a well defined cup product*

$$H^m(X) \times H^n(X) \xrightarrow{\cup} H^{m+n}(X).$$

Proof. $Z^m(X) \times Z^n(X) \xrightarrow{\cup} Z^{m+n}(X)$. If $\phi \in Z^m(X)$, $\phi \cup \delta\psi = (-1)^k \delta(\phi \cup \psi)$. Similarly $B^m(X) \times Z^n(X) \xrightarrow{\cup} B^{m+n}(X)$. □

If our $\phi \in S^m(X, A) \subset S^m(X)$, then $\phi \cup \psi \in S^{m+n}(X, A)$. Thus we have

$$H^m(X, A) \times H^n(X) \xrightarrow{\cup} H^{m+n}(X, A)$$

$$H^m(X) \times H^n(X, A) \xrightarrow{\cup} H^{m+n}(X, A)$$

If $A, B \subset X$ are relatively open subsets of $A \cup B$. Then one can define

$$H^m(X, A) \times H^n(X, B) \xrightarrow{\cup} H^{m+n}(X, A \cup B)$$

as follows. At cochain level, take $\phi \in S^m(X, A)$ and $\psi \in S^n(X, B)$, we have clearly $\phi \cup \psi \in S^{m+n}(X, A) \cap S^{m+n}(X, B)$. However, it is not $S^{m+n}(X, A \cup B)$. So we need the fact that the inclusion $S_*(A) + S_*(B) \subset S_*(A \cup B)$ induces isomorphism in homology, which is proved in Algebraic Topology module (Proposition 2.21 of Hatcher). This tells us cochain complex $S^*(A \cup B, A) \cap S^*(A \cup B, B)$ has trivial cohomology.

Look at the short exact sequence

$$0 \to S^*(X, A \cup B) \to S^*(X, A) \cap S^*(X, B) \to S^*(A \cup B, A) \cap S^*(A \cup B, B) \to 0.$$

We know the inclusion $S^*(X, A \cup B) \to S^*(X, A) \cap S^*(X, B)$ induces isomorphism on cohomology.

Lemma 2.1.4. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y; R) \to H^n(X; R)$ satisfy $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$, and similarly in the relative case.*

Proof. Let α and β represented by $\phi \in S^m$ and $\psi \in S^n$ respectively. Let $\sigma : \Delta^{m+n} \to X$. So $f(\sigma)$ is a $(m + n)$-simplex in Y. Then

$$\begin{aligned} f^*(\phi \cup \psi)(\sigma) =&(\phi \cup \psi)(f(\sigma)) \\ =&\phi(f(\sigma)|[v_0, \cdots, v_m])\psi(f(\sigma)|[v_m, \cdots, v_{m+n}]) \\ =&f^*(\phi)(\sigma|[[v_0, \cdots, v_m])f^*\psi(\sigma|[v_m, \cdots, v_{m+n}) \\ =&f^*(\phi) \cup f^*(\psi)(\sigma) \end{aligned}$$

□

It is easy to check that the cup product is associative (even at chain level): $(\phi \cup \psi) \cup \theta = \phi \cup (\psi \cup \theta)$. We also know there is a unit: Let $1 \in S^0(X)$ be the function takes value 1 on any point of X, then $1 \cup a = a$.

Proposition 2.1.5. *If $\alpha \in H^m(X;R)$ and $\beta \in H^n(X;R)$ then*

$$\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha.$$

Proof. For a singular n-simplex $\sigma : [v_0, \cdots, v_n] \to X$, let $\bar{\sigma} = \sigma \circ r$ where r is the linear map determined by $r(v_i) = v_{n-i}$. Then we define $\rho : S_n(X) \to S_n(X)$ by $\rho(\sigma) = \epsilon_n \bar{\sigma}$ where $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$.

We want to show ρ is a chain map which is chain homotopic to the identity. Once we have that, the theorem follows:

$$(\rho^*\phi \cup \rho^*\psi)(\sigma) = \phi(\epsilon_m \sigma|[v_m, \cdots, v_0])\psi(\epsilon_n \sigma|[v_{m+n}, \cdots, v_m])$$

$$\rho^*(\psi \cup \phi)(\sigma) = \epsilon_{m+n}\psi(\sigma|[v_{m+n}, \cdots, v_m])\phi(\sigma|[v_m, \cdots, v_0])$$

with $\epsilon_{m+n} = (-1)^{mn}\epsilon_m\epsilon_n$, show that

$$\rho^*\phi \cup \rho^*\psi = (-1)^{mn}\rho^*(\psi \cup \phi)$$

at chain level. Since ρ induces identity in cohomology, we have $\alpha \cup \beta = (-1)^{mn}\beta \cup \alpha$ when passing to the cohomology level.

Now the proof reduces to the following two lemmas. □

Lemma 2.1.6. *ρ is a chain map.*

Proof. For an n-simplex σ,

$$\partial\rho(\sigma) = \epsilon_n \sum_i (-1)^i \sigma|[v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0]$$

$$\begin{aligned}\rho\partial(\sigma) =& \rho(\sum_i (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]) \\ =& \epsilon_{n-1} \sum_i (-1)^{n-i} \sigma|[v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0]\end{aligned}$$

They are equal since $\epsilon_n = (-1)^n \epsilon_{n-1}$. □

Lemma 2.1.7. *This chain map ρ is chain homotopic to the identity and so it induces the identity homomorphism in cohomology.*

Proof. We need the fact that there is a natural division of $[v_0, \cdots, v_n] \times I$ into $n+1$ simplices. If we denote $(v_i, 0)$ by v_i and $(v_i, 1)$ by w_i, then these simplices are

$$\sigma_i = [v_0, \cdots, v_i, w_i, \cdots, w_n].$$

I.e. trace out along the bottom until position i, then jump to top, trace out the rest start with position i.

We define $P : S_n(X) \to S_{n+1}(X)$ by

$$P(\sigma) = \sum_i (-1)^i \epsilon_{n-i} (\sigma \circ \pi)|[v_0, \cdots, v_i, w_n, \cdots, w_i],$$

where $\pi : \Delta \times I \to \Delta$ is a projection. We will leave out $\sigma \circ \pi$ for notational simplicity.

$$\begin{aligned}\partial P(\sigma) =& \sum_{j\le i}(-1)^i(-1)^j\epsilon_{n-i}[v_0,\cdots,\hat{v_j},\cdots,v_i,w_n,\cdots,w_i]\\ &+\sum_{j\ge i}(-1)^i(-1)^{i+1+n-j}\epsilon_{n-i}[v_0,\cdots,v_i,w_n,\cdots,\hat{w_j},\cdots,w_i]\\ P\partial(\sigma) =& \sum_{i<j}(-1)^i(-1)^j\epsilon_{n-i-1}[v_0,\cdots,v_i,w_n,\cdots,\hat{w_j},\cdots,w_i]\\ &+\sum_{i>j}(-1)^{i-1}(-1)^j\epsilon_{n-i}[v_0,\cdots,\hat{v_j},\cdots,v_i,w_n,\cdots,w_i]\end{aligned}$$

Since $\epsilon_{n-i} = (-1)^{n-i}\epsilon_{n-i-1}$, the terms with $i \neq j$ cancel in the two sums. The terms with $j = i$ give

$$\begin{aligned}\epsilon_n[w_n,\cdots,w_0]+&\sum_{i>0}\epsilon_{n-i}[v_0,\cdots,v_{i-1},w_n,\cdots,w_i]\\ &+\sum_{i<n}(-1)^{n+i+1}\epsilon_{n-i}[v_0,\cdots,v_i,w_n,\cdots,w_{i+1}]-[v_0,\cdots,v_n]\end{aligned}$$

The two summations cancel, as replacing i by $i-1$ in the second sum produces a new sign $(-1)^{n+i}\epsilon_{n-i+1} = -\epsilon_{n-i}$. Thus the remaining two terms is just

$$\partial P(\sigma) + P\partial(\sigma) = \epsilon_n[w_n,\cdots,w_0] - [v_0,\cdots,v_n] = \rho(\sigma) - \sigma.$$

Hence P is the chain homotopy relates ρ to identity. □

Remark 2.1.8. *As we mentioned in the class, this proof should be read using picture.*

P is the map from a simplex to get an oriented cylinder with simplicial division. Then ∂P is the oriented boundary of the cylinder. $P\partial$ is the part of the boundary without top and bottom and with opposite sign.

Cancellations could also be interpreted: the cancellation on $i \neq j$ is the cancellation on the boundary without top and bottom. Then second cancellation above is the intersection face of different simplices in the division with different orientation.

To summarize, we have

Theorem 2.1.9 (The cohomology ring)**.** *Let X be a topological space and R a commutative ring with identity. Then*

$$H^*(X;R) = \bigoplus_i H^i(X;R)$$

is a graded commutative ring with identity under the cup product, i.e. if $\alpha \in H^m(X;R)$ and $\beta \in H^n(X;R)$ then $\alpha \cup \beta = (-1)^{mn}\beta \cup \alpha$. Moreover $H^*(X;R)$ is an R-algebra.

Example 2.1.10. *$H^*(S^n;\mathbb{Z}) = \frac{\mathbb{Z}[a_n]}{a_n^2}$, where the generator of H^0 corresponds to 1 and the generator of H^n is a_n.*

Actually, we are ready to calculate the cohomology ring of projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$. But I would like to leave these after Poincaré duality.

We remark that all the above results extend to relative case, i.e replacing $H^*(X;R)$ with $H^*(X,A;R)$.

2.1.1 Cap product

For any space X, there is a bilinear pairing operation between cochains and chains.

Definition 2.1.11. *Let $a \in S^q(X)$ and $\sigma \in S_{p+q}(X)$. Then the cap product $a \cap \sigma \in S_p(X)$ is*

$$a \cap \sigma =< a, \sigma|[v_p, \cdots, v_{p+q}] > \cdot \sigma|[v_0, \cdots, v_p], \text{ or } a \cap \sigma =< a, \sigma_q > \cdot_p \sigma.$$

The cap product at chain level has the following properties.

Proposition 2.1.12. *1. Duality: For $a, b \in S^*(X)$, $c \in S_*(X)$, we have*

$$< a \cup b, c >=< a, b \cap c > .$$

2. Associativity: For $a, b \in S^*(X)$, $c \in S_*(X)$, we have

$$(a \cup b) \cap c = a \cap (b \cap c).$$

3. Unit:

$$1 \cap c = c.$$

4. Naturality: Let $f : X \to Y$ be a map. For $b \in S^*(Y)$ and $c \in S_*(X)$, we have

$$b \cap (f_*c) = f_*(f^*b \cap c).$$

All these are easy to derive from the definition and the properties of cup product.

To define cap product on (co)homology, we need

Proposition 2.1.13. *For $a \in S^q(X)$ and $\sigma \in S_{p+q}(X)$, we have*

$$\partial(a \cap \sigma) = (-1)^p(\delta a) \cap \sigma + a \cap (\partial\sigma).$$

Proof. For simplicity, we omit the notation σ in the calculation.

$$\begin{aligned}
a \cap \partial[v_0, \cdots, v_{p+q}] =& \sum_{i=0}^{p+q} (-1)^i a \cap [v_0, \cdots, \hat{v}_i, \cdots, v_{p+q}] \\
=& \sum_{i=0}^{p} (-1)^i < a, [v_p, \cdots, v_{p+q}] > [v_0, \cdots, \hat{v}_i, \cdots, v_p] \\
&+ (-1)^{p-1} \sum_{i=p-1}^{p+q} (-1)^{i-p+1} < a, [v_{p-1} \cdots, \hat{v}_i, \cdots, v_{p+q}] > [v_0, \cdots, v_{p-1}] \\
=& \partial(a \cap \sigma) + (-1)^{p-1} < a, \partial[v_{p-1} \cdots, v_{p+q}] > [v_0, \cdots, v_{p-1}] \\
=& \partial(a \cap \sigma) + (-1)^{p-1} (\delta a) \cap \sigma
\end{aligned}$$

□

Hence it induces the cap product between homology and cohomology:

$$H^q(X) \times H_{p+q}(X) \xrightarrow{\cap} H_p(X).$$

2.1.2 De Rham cohomology

Let M be a smooth manifold of dimension n now. Let $\Omega^q(M)$ be the (real linear) space of q-forms, let $d : \Omega^q(M) \to \Omega^{q+1}(M)$ be the exterior differential. Then we have the de Rham cochain complex

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

We denote he space of closed (exact) k-forms, i.e k-forms ω with $d\omega = 0$ ($\omega = d\eta$ respectively), by $Z^k_{dR}(M)$ (and $B^k_{dR}(M)$ respectively). We denote its cohomology by $H^*_{dR}(M)$, which is called the de Rham cohomology of M.

The cup product for de Rham cohomology is just the *wedge product* $\omega \wedge \eta$. Since for k-form ω

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

we know it descends to a product on cohomology by the same argument as Corollary 2.1.3. A notable fact for de Rham cohomology is its cup product is graded commutative even at chain level: $x \wedge y = (-1)^{|x||y|} y \wedge x$.

There are several reasons prevent us from exploiting the Eilenberg-Steenrod uniqueness theorem to claim it is isomorphic to singular cohomology $H^*(X; \mathbb{R})$. The main reason is de Rham cohomology is only defined for smooth manifolds. The lack of relative version of de Rham theory could be compensated by Thom isomorphism.

However, de Rham theorem does ensure this isomorphism.

Theorem 2.1.14 (de Rham theorem)**.** *$H^*_{dR}(M) = H^*(M;\mathbb{R})$ as cohomology rings.*

More precisely, we need two facts for this theorem:
1. Let $S_q^{sm}(M;\mathbb{R})$ be the real space spanned by smooth singular q-simplices $\sigma : \Delta^q \to M$, then the inclusion

$$S_*^{sm}(M;\mathbb{R}) \to S_*(M;\mathbb{R})$$

is a chain homotopic equivalence. Then its dual

$$S^*(M;\mathbb{R}) \to S^*_{sm}(M;\mathbb{R})$$

is a cochain homotopic equivalence.
2. We could take integration of a q-form on a singular chain of dimension q, this is a bilinear function

$$\Omega^q(M) \times S_q^{sm}(M;\mathbb{R}) \to \mathbb{R}, (\omega, \sigma) \mapsto \int_\sigma \omega.$$

Stokes theorem tells us

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega.$$

In other words, exterior differential is dual to boundary map. This provides us a cochain map

$$\Omega^*(M) \to S^*_{sm}(M;\mathbb{R}).$$

If we show this is a cochain homotopic equivalence, we finish the proof of de Rham theorem. The proof proceeds as the same pattern as the proof of Poincaré duality which we will provide in next chapter.

2.2 Cross Product and Künneth formula

We want to understand the cohomology ring of a product space. Let us first define the cross product in cohomology.

Definition 2.2.1. *If $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections, then define the cross product by*

$$x \times y = p_1^*(x) \cup p_2^*(y) \in H^{m+n}(X \times Y; R)$$

for $x \in H^m(X;R)$ and $y \in H^n(Y;R)$.

To be more precise, for $\sigma : \Delta^{m+n} \to X \times Y$, let $\sigma_1 = p_1 \circ \sigma : \Delta^{m+n} \to X$ and $\sigma_2 = p_2 \circ \sigma : \Delta^{m+n} \to Y$. Then for $\phi \in S^m(X)$ and $\psi \in S^n(Y)$, we have

$$(\phi \times \psi)(\sigma) = \phi(\sigma_1|[v_0, \cdots, v_m])\psi(\sigma_2|[v_m, \cdots, v_{m+n}])$$

We could also extend this to a relative version

$$H^m(X, A) \times H^n(Y, B) \to H^{m+n}(X \times Y, X \times B \cup A \times Y).$$

Since $\times$ is bilinear, it factors through the tensor product to give a linear map (also denoted $\times$)

$$H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R).$$

Proposition 2.2.2. *For any $a, c \in H^*(X; R)$ and $b, d \in H^*(Y; R)$ with $c \in H^m(X; R)$ and $b \in H^n(Y; R)$ we have*

$$(a \times b) \cup (c \times d) = (-1)^{mn}(a \cup c) \times (b \cup d).$$

Proof. Because $(p_1^* a \cup p_2^* b) \cup (p_1^* c \cup p_2^* d) = (-1)^{mn} p_1^*(a \cup c) \cup p_2^*(b \cup d)$. □

Exercise: By above proposition and induction for dimension n, prove the cohomology ring $H^*(T^n)$ for a n-torus is isomorphic to exterior algebra $\Lambda_{\mathbb{Z}}[x_1, \cdots, x_n]$. Any generator has dimension 1.

Recall that the exterior algebra $\Lambda_R[x_1, x_2 \cdots]$ over a commutative ring R with identity is the free R-module with basis the finite product $x_{i_1} \cdots x_{i_k}, i_1 < \cdots < i_k$, with the multiplication defined by the rules $x_i x_j = -x_j x_i$ and $x_i^2 = 0$.

The following Künneth formula generalizes the previous example.

Theorem 2.2.3. *The cross product $H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y; R)$ is a finitely generated free R-module for all k.*

We want to use Theorem 1.6.1. We show the theorem for finite CW complexes. For general CW complexes, as we mentioned before, we need Milnor additivity axiom. Consider the following functors:

$$h^n(X, A) = \oplus_{i+j}(H^i(X, A; R) \otimes_R H^j(Y; R),$$

$$k^n(X, A) = H^n(X \times Y, A \times Y; R).$$

We have $\phi : h^n(X, A) \to k^n(X, A)$ given by the cross product. So we need to check

1. h^* and k^* are cohomology theories.
2. ϕ is a natural transformation.

Proof. First we check h^* and k^* are cohomology theories. All axioms are easy to verify. A few words for exactness axiom. The exactness for k^* is trivial. For h^*, it is where we use the freeness of $H^k(Y; R)$.

Naturality of ϕ with respect to maps between spaces is from the naturality of cup products. Natuarality with respect to the coboundary maps is to check the following diagram commutes. We save R in our notation.

$$\begin{array}{ccc} H^k(A)\times H^l(Y) & \xrightarrow{\delta\times id} & H^{k+1}(X,A)\times H^l(Y) \\ \Big\downarrow\times & & \Big\downarrow\times \\ H^{k+l}(A\times Y) & \xrightarrow{\delta} & H^{k+l+1}(X\times Y, A\times Y) \end{array}$$

To check this, start with an element (a,b) represented by cocycle $\phi \in S^k(A)$ and $\psi \in S^l(Y)$. Extend ϕ to a cochain $\bar{\phi} \in S^k(X;R)$. Then (ϕ,ψ) maps rightward to $(\delta\bar{\phi},\psi)$ and then downward to $\pi_X^*(\delta\bar{\phi}) \cup \pi_Y^*(\psi)$. On the other direction, (ϕ,ψ) maps downward to $\pi_X^*(\phi)\cup\pi_Y^*(\psi)$ and then rightward to $\delta(\pi_X^*(\bar{\phi}) \cup \pi_Y^*(\psi)) = \pi_X^*(\delta\bar{\phi}) \cup \pi_Y^*(\psi)$.

I hope the chain level δ and connecting homomorphism δ do not confuse you! □

Example 2.2.4. *Now it is more straightforward to show*

$$H^*(T^n) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_n].$$

Similarly, one could also show

$$H^*(S^n \times S^m) = \frac{[a_n, a_m]}{a_n^2 = 0, a_m^2 = 0}.$$

2.2.1 An alternative way to define cross and cup products

We could construct cup product for CW complexes. By virtue of Eilenberg-Steenrod uniqueness, this cup product is the same as that of singular cohomology.

The cross product in this setting is quite natural. Start with that of chain level. Take cells $e^i \in X$ and $e^j \in Y$, then we could send it to product cell $e^i \times e^j$ in $X \times Y$. One could extend this map by tensor product from $C_*(X) \otimes C_*(Y)$ to $C_*(X \times Y)$. Then for a pair of cocycles z_1, z_2 of X and Y, it thus yields a cocycle $z_1 \times z_2$. This defines

$$H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y).$$

One could check it is the same as our previous defined cross product.

Then by using the diagonal map $\Delta : X \to X \times X, x \mapsto (x,x)$, we can define cup product as the composition

$$H^k(X) \times H^l(X) \xrightarrow{\times} H^{k+l}(X \times X) \xrightarrow{\Delta^*} H^{k+l}(X).$$

This is our previous cup product since

$$\Delta^*(a \times b) = \Delta^*(p_1^*(a) \cup p_2^*(b)) = a \cup b.$$

But unfortunately, this cup product is not defined at the level of cellular cochains, which prevents us to prove the properties. To resolve this issue, one need to find cellular map which is homotopic to Δ. This is in general true for maps between CW complexes. And for our case $X \to X \times X$, the map is actually a slight modification of P (called Alexander-Whitney chain approximation) used in the proof of graded commutativity of cup product. We do not go through the detail again.

2.3 Ljusternik-Schnirelmann category

Definition 2.3.1. *The Ljusternik-Schnirelmann category $cat(X)$ of a topological space X is defined to be the smallest integer k such that there is an open covering $\{U_i\}_{1\le i\le k}$ of X such that each inclusion $U_i \hookrightarrow X$ is null-homotopic (we call U_i is contractible in X), i.e homotopic to a constant map.*

Example 2.3.2. *Notice the subtly of the definition. S^{n-1} is contractible in D^n, although S^{n-1} itself is not contractible space.*

Example 2.3.3. $cat(S^1) = 2$. *Actually, for any suspension $\Sigma(X)$, $cat(\Sigma(X)) \le 2$ since it could be covered by two contractible sets $C_+(X)$ and $C_-(X)$.*

Notice $cat(M) < \infty$ if M is a compact manifold since it could be covered with finitely many sets homeomorphic to open discs. And in fact $cat(M) \le \dim M + 1$. In general, there is no reason for $cat(X)$ to be finite.

Definition 2.3.4. *The cup length $cl(X)$ of a topological space X is defined to be*

$$cl(X) := \max\{n | \text{there exist } \alpha_i \in H^{m_i}(X) \text{ with } m_i > 0 \text{ such that } \alpha_1 \cup \cdots \cup \alpha_n \neq 0\}.$$

Proposition 2.3.5. *For any space X, we have $cl(X) < cat(X)$.*

Proof. Suppose $cat(X) = n$, so $X = \bigcup_{j=1}^n U_j$ with each U_k contractible in X. We denote the inclusion by i_k. Since i_k is nullhomotopic, its induced homomorphism i_k^* could be decomposed as $H^*(X) \to H^*(pt) \to H^*(U_k)$. So when $q > 0$, $i_k^* = 0 : H^q(X) \to H^q(U_k)$. By the cohomology exact sequence for the pair (X, U_k), we know $j_k^* : H^q(X, U_k) \to H^q(X)$ is surjective. So for any $\xi_k \in H^*(X)$, we have $\eta_k \in H^*(X, U_k)$ such that $\xi_k = j_k^*(\eta_k)$.

Look at the commute diagram

$$\begin{array}{ccccccc}
H^*(X,U_1) & \times & \cdots & \times & H^*(X,U_n) & \xrightarrow{\cup} & H^*(X,\cup_{k=1}^n U_k) \\
\downarrow j_1^* & & & & \downarrow j_n^* & & \downarrow j^* \\
H^*(X) & \times & \cdots & \times & H^*(X) & \xrightarrow{\cup} & H^*(X)
\end{array}$$

Hence $\xi_1 \cup \cdots \cup \xi_n = j_1^*(\eta_1) \cup \cdots \cup j_n^*(\eta_n) = j^*(\eta_1 \cup \cdots \cup \eta_n) = 0$. The last equality is because of $H^*(X, \cup_{k=1}^n U_k) = H^*(X, X) = 0$. □

Example 2.3.6. *An n-torus T^n has $cat(T^n) = n + 1$.*

Example 2.3.7. *So for any suspension, $cl(\Sigma(X)) = 1$ if X is not weakly contractible. This tells us cup product does not commute with suspension and hence not a stable property.*

Ljusternik-Schnirelmann use the notion of *cat* to study critical points. Their main theorem is the following

Theorem 2.3.8. *Let M be a smooth connected compact manifold and $f : M \to \mathbb{R}$ is a smooth function. Then f has at least $cat(X)$ critical points.*

Example 2.3.9. *Any smooth function on T^2 has at least 3 critical points. Can you construct a smooth function on T^2 with exactly 3 critical points? (Hint: use the viewpoint in the proof of the following theorem, construct a vector field on torus with 3 singular points.)*

Actually, Ljusternik-Schnirelmann category is an example of general category (It has nothing to do with "category theory"). We assume X be a locally contractible path connected space.

Definition 2.3.10. *A category is an assignment $\nu : 2^X \to \mathbb{N} \cup \{0\}$ (where 2^X denotes the set of all subsets in X, i.e. the power set) satisfying the following axioms:*

- *Continuity: for every $A \in 2^X$ there exists an open set $U \supset A$ such that $\nu(A) = \nu(U)$.*
- *Monotonicity: if $A, B \in 2^X$ with $A \subset B$ then $\nu(A) \leq \nu(B)$.*
- *Subadditivity: for any $A, B \in 2^X$ we have $\nu(A \cup B) \leq \nu(A) + \nu(B)$.*
- *Naturality: if $\phi : X \to Y$ is a homeomorphism then for any $A \in 2^X$, $\nu_Y(\phi(A)) = \nu_X(A)$.*
- *Normalization: $\nu(\emptyset) = 0$, and if $A = \{x_0, \cdots, x_n\}$ is a finite set then $\nu(A) = 1$.*

To prove Theorem 2.3.8, we prove the following more general proposition.

Proposition 2.3.11. *Let X be a locally contractible path connected compact metric space, and ϕ_t a global flow on X. Suppose there exists a Lyapunov function $\Phi : X \to \mathbb{R}$ such that Φ strictly decreases along non-constant orbits of ϕ_t. Then Φ has at least $\nu(X)$ critical points where ν is any category.*

Proof. Let $X^c := \Phi^{-1}(-\infty, c]$. A critical value for Φ is a value such that $\Phi^{-1}(c)$ contains a constant orbit. If c is not critical, then for sufficiently small $\delta > 0$, we could find $t > 0$ such that

$$\phi_t(X^{c+\delta}) \subset X^{c-\delta},$$

since Φ strictly decreases away from the constant orbits. If c is a critical level and U is a neighorhood of $\Phi^{-1}(c)$ then for small $\delta > 0$, we have $t > 0$

$$\phi_t(X^{c+\delta} \backslash U) \subset X^{c-\delta}.$$

By naturality and monotonicity, $\nu(X^{c+\delta} \backslash U) \leq \nu(X^{c-\delta})$.

For $j = 1, \cdots, N = \nu(X)$, let

$$c_j := \sup\{c | \nu(X^c) < j\}.$$

Then $c_1 = \min\{\Phi\}$ and $c_N = \max\{\Phi\}$. Note that c_j is a critical value of Φ for each j.

Now we want to prove either $c_j < c_{j+1}$ or $\Phi^{-1}(c_j)$ contains infinitely many critical points. If the latter happens, the theorem follows immediately. So we assume the latter does not happen, say $\Phi^{-1}(c_j) = \{x_0, \cdots, x_n\}$. Then by continuity axiom, there exists a neighborhood U of $\{x_0, \cdots, x_n\}$ such that $\nu(U) = 1$.

Then by subadditivity, we have

$$\begin{aligned} \nu(X^{c_j+\delta}) \leq & \nu(X^{c_j+\delta} \backslash U) + 1 \\ \leq & \nu(X^{c_j-\delta}) + 1 \\ \leq & j \end{aligned}$$

Hence $c_{j+1} \geq c_j + \delta > c_j$. Thus $c_1 < \cdots < c_N$ are $N = \nu(X)$ different critical points. This completes the proof. □

Then let us finish the proof of Theorem 2.3.8: Give M a Riemannian metric and let ∇f denote the gradient of f with respect to this metric, i.e. it is the unique vector field determined by

$$< (\nabla f)_p, V_p >= df_p(V_p)$$

for every vector field V. The critical points of f are precisely the zeros of ∇f. Let ϕ_t be the associated flow of ∇f, i.e.

$$\frac{d\phi_t(p)}{dt} = -(\nabla f)_{\phi_t(p)}.$$

We claim that f is a Lyapunov function for ϕ_t:

$$\begin{aligned}\frac{d}{dt}(f\circ\phi_t(p)) =& df(\frac{d\phi_t(p)}{dt})\\ =& < (\nabla f)_{\phi_t(p)}, \frac{d\phi_t(p)}{dt} >\\ =& - < \frac{d\phi_t(p)}{dt}, \frac{d\phi_t(p)}{dt} >\\ \leq& 0\end{aligned}$$

with equality holds if and only if $\frac{d\phi_t(p)}{dt} = 0$, i.e. p is a critical point of f, and $\phi_t(p)$ is a constant orbit p. This completes the proof.

2.4 Higher products

Later we will see that the linking number of two spheres S^p and S^q in R^{p+q+1} will be understood as cup product of the cohomolgy ring of the complement $H^*(R^{p+q+1}\backslash(S^p \cup S^q))$. There are links with 3 components with each two of them are unlinked, but nonetheless all three are link. The most famous example is the Borromean rings. This more complicated linking phenomenon for three or more spheres suggest the existence of higher cup product: the Massey product. We start with Massey triple product.

Assume $[u], [v], [w]$ are cohomology classes of dimension p, q and r respectively, represented by $u \in Z^p(X)$, $v \in Z^q(X)$ and $w \in Z^r(X)$. If $[u] \cup [v] = 0 = [v] \cup [w]$, then we introduce a (set of) new cohomology classes. For the notation, we introduce $\bar{u} = (-1)^{1+\deg u}u$.

Since $[u]\cup[v] = 0$, we have $s \in C^{p+q-1}(X)$ such that $\delta s = \bar{u}\cup v$. Similarly, we have $t \in C^{q+r-1}(X)$ such that $\delta t = \bar{v} \cup w$. The element $\bar{s} \cup w + \bar{u} \cup t$ determines a cocycle in $Z^{p+q+r-1}(X)$:

$$\begin{aligned}\delta(\bar{s} \cup w + \bar{u} \cup t) =& (-1)^{p+q}\delta s \cup w + (-1)^p \bar{u} \cup \delta t\\ =& (-1)^{p+q}\bar{u} \cup v \cup w + (-1)^{p+q+1}\bar{u} \cup v \cup w = 0\end{aligned}$$

We define the *Massey triple product* as the set of all such cohomology classes

$$< [u], [v], [w] >= \{[\bar{s} \cup w + \bar{u} \cup t] | \delta s = \bar{u} \cup v, \delta t = \bar{v} \cup w\}.$$

There are indeterminacy from the different choices of representatives. But we could identify them as following.

Proposition 2.4.1. *The Massey triple product* $< [u], [v], [w] >$ *is an element of quotient group* $H^{p+q+r-1}(X)/([u] \cup H^{q+r-1}(X) + H^{p+q-1}(X) \cup [w])$.

Proof. We need to show different choices of u, v, w, s, t do not affect the coset in $H^{p+q+r-1}(X)$ given above. We only check that of s. Others are left as exercise.

If s and s' are chosen such that $\delta s = \bar{u} \cup v = \delta s'$, then

$$(\bar{s} \cup w + \bar{u} \cup t) - (\bar{s}' \cup w + \bar{u} \cup t) = (\bar{s} - \bar{s}') \cup w,$$

which is in $H^{p+q-1}(X) \cup [w]$ as cohomology classes. □

The Massey product was used to prove the Jacobi identity for Whitehead product in homotopy group.

We could define higher order Massey products. When two triple products $< [u], [v], [w] >$ and $< [v], [w], [x] >$ are defined, and if $0 \in < [u], [v], [w] >$ and $0 \in < [v], [w], [x] >$, then we could find

$$\delta Y_1 = \bar{t}_0 \cup w + \bar{u} \cup t_1, \delta Y_2 = \bar{t}_1 \cup x + \bar{v} \cup t_2$$

where $\delta t_0 = \bar{u} \cup v, \delta t_1 = \bar{v} \cup w, \delta t_2 = \bar{w} \cup x$. Then we form a subset $< [u], [v], [w], [x] >$ in $H^{|u|+|v|+|w|+|x|-2}(X)$ whose elements are

$$\bar{u} \cup Y_2 + \bar{t}_0 \cup t_2 + \bar{Y}_1 \cup x.$$

This is called a fourfold product.

As we mentioned in the class, it is better to understand the whole picture by matrix

$$\begin{pmatrix} u & s & \\ & v & t \\ & & w \end{pmatrix}$$

$$\begin{pmatrix} u & t_0 & Y_1 & \\ & v & t_1 & Y_2 \\ & & w & t_2 \\ & & & x \end{pmatrix}$$

We can do inductively to define n-fold Massey product. $< [a_{1,1}], [a_{2,2}], \cdots, [a_{n,n}] >$ to be the set of elements of the forms

$$\bar{a}_{1,1} a_{2,n} + \bar{a}_{1,2} a_{3,n} + \cdots + \bar{a}_{1,n-1} a_{n,n}$$

for all solutions of the equations

$$\delta a_{i,j} = \bar{a}_{i,i} a_{i+1,j} + \bar{a}_{i,i+1} a_{i+2,j} + \cdots + \bar{a}_{i,j-1} a_{j,j}, 1 \le i \le j \le n, (i,j) \ne (1,n).$$

Hence, to ensure the set is non-empty, we need the vanishing of many lower order Massey product.

We remark that the Massey products are defined for (homology) of a *differential graded algebra* (DGA) A. It is a graded algebra $A = \oplus_{k \le 0} A^k$ with a differential $d : A \to A$ of degree $+1$, such that

1. A is graded commutative, i.e

$$x \cdot y = (-1)^{kl} y \cdot x, x \in A^k, y \in A^l$$

2. d is a derivation, i.e.

$$d(x \cdot y) = dx \cdot y + (-1)^k x \cdot dy, x \in A^k$$

3. $d^2 = 0$

Examples include cohomology ring with 0 as its differential and de Rham complex (Ω^*, d) on a manifold.

Rational homotopy theory of Quillen and Sullivan is built to understand (real) homotopy group by the structure of DGA. For example, a manifold on which all Massey products vanish is a formal manifold: its real homotopy type follows ("formally") from its real cohomology ring. Deligne-Griffiths-Morgan-Sullivan proved that all Kähler manifolds are formal.

Chapter 3

Poincaré duality

We want to prove the Poincaré duality in this chapter. For a compact n-manifold M, this asserts that $H^p(M^n)$ is isomorphic to $H_{n-p}(M^n)$. It is the most important result in this course, and has lots of important applications. Poincaré's original proof used the idea of dual cell structures (explained in the class by pictures). It is rather intuitive and geometric proof. However, we will lose some generality if we use this method.

Hence, we use the proof of Milnor. The basic idea of Milnor's proof is very natural and explained as follows. We know that any n-manifold is a union of open subsets, each of which is homeomorphic to $\mathbb{R}^n$. It is natural to first prove (certain version of) the theorem for $\mathbb{R}^n$, and then use Mayer-Vietoris sequences to prove the case of a finite union of open subsets. Finally, it passes to the case of an infinite union by a direct limit argument. We will then state and prove a more general version which is applicable to noncompact manifolds since we have to first deal with $\mathbb{R}^n$. For this reason, we need to introduce cohomology with compact supports.

3.1 Cohomology with compact supports

Suppose M is a topological space. The *singular cochains with compact support* is defined as $\alpha \in S^p(M)$ such that there is a compact set $K \subset M$ such that $\alpha \in S^p(M, M\backslash K) \subset S^p(M)$, i.e. $\alpha|_{M\backslash K} = 0$. Write $S_c^p(M)$ for the set of all these cochains. Note that δ preserves $S_c^*(M)$. Hence

Definition 3.1.1. $H_c^p(M) := H^p(S_c^*(M))$ *is the cohomology with compact support of* M.

Observe that if M is compact $H_c^*(M) = H^*(M)$. We will need to calculate $H_c^*(M)$ in general for the proof of Poincaré duality. We want to understand it by relative cohomology groups. So we introduce the definition of a direct limit.

Definition 3.1.2. *A directed system of abelian groups* $\{G_\alpha | a \in A\}$ *is a collection of abelian groups indexed by a partially ordered set* A *satisfying*

1. *For all* $a, b \in A$ *there exists* $c \in A$ *such that* $a \leq c$ *and* $b \leq c$.

2. *For all* $a \leq b$ *there exists a homomorphism* $f_{ab} : G_a \to G_b$ *such that* $f_{aa} = id$ *and if* $a \leq b \leq c$, $f_{ac} = f_{bc} \circ f_{ab}$.

Recall that a partially ordered set (or poset) is a set A along with a binary relation $\leq$ which is reflexive, antisymmetric and transitive:

1. Reflexive: $a \leq a$;

2. Antisymmetric: if $a \leq b$ and $b \leq a$, then $a = b$;

3. Transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$.

A partially ordered set with property 1 in Definition 3.1.2 is called a *directed set*. The main example of directed sets in our mind is the set of compact subsets with partial ordering $\subset$ and f is the inclusion.

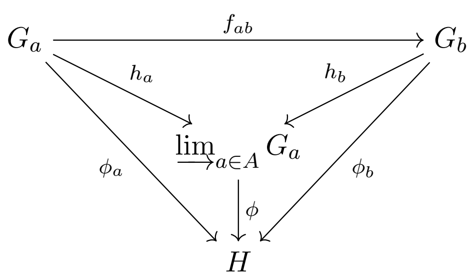
Definition 3.1.3. *Given a direct system* $\{G_\alpha | a \in A\}$, *the Direct Limit (colimit) is defined to be*

$$\varinjlim_{a \in A} G_a := (\oplus_{a \in A} G_a) \backslash N$$

where $N \subset \oplus_{a \in A} G_a$ *is the subgroup generated by* $x - f_{ab}(x)$ *where* $x \in G_a$ *and* $a \leq b$.

Direct limit has the following universal property. Actually, this property characterizes direct limit.

Proposition 3.1.4 (Universal property of direct limit)**.** *Any homomorphisms* $\phi_a : G_a \to H$ *such that* $\phi_a = \phi_b \circ f_{ab}$ *for any pair* $a \leq b$ *factor through* $\varinjlim_{a \in A} G_a$. *That is, there exists a unique homomorphism* $\phi : \varinjlim_{a \in A} G_a \to H$ *such that* $\phi \circ h_a = \phi_a$ *for all* $a \in A$. *Here* $h_a : G_a \to \varinjlim_{a \in A} G_a$ *are the inclusion maps.*



Proof. For any $g \in G_a$, we denote its equivalence class in $\varinjlim_{a\in A} G_a$ as $[g]$. So $h_a(g) = [g]$.

Let us first construct a ϕ: $\phi([g]) = \phi_a(g)$. It is easy to check that it makes the diagram commutes. Let us check it is well defined. Suppose there are $g_1 \in G_a$ and $g_2 \in G_b$ such that $[g_1] = [g_2]$. Then we know there is a c such that $a \leq c$ and $b \leq c$, and $f_{ac}(g_1) = f_{bc}(g_2)$. Then we see that

$$\phi_a(g_1) = \phi_c \circ f_{ac}(g_1) = \phi_c \circ f_{bc}(g_2) = \phi_b(g_2).$$

Then we prove the uniqueness. Otherwise there is another $\phi' : \varinjlim_{a\in A} G_a \to H$ such that $\phi' \circ h_a = \phi_a$ for all $a \in A$. Then

$$\phi'([g]) = \phi' \circ h_a(g) = \phi_a(g) = \phi([g]).$$

In other words, $\phi' = \phi$. □

Now we could give an alternative definition of $H_c^*(M)$ in terms of direct limit. The compact subsets $K \subset M$ form a directed set under inclusion. For $K \subset L$, we have the inclusion $(M, M\backslash L) \to (M, M\backslash K)$, and thus the homomorphism $H^p(M, M\backslash K) \to H^p(M, M\backslash L)$. Hence we have the direct limit

$$\varinjlim_{K\subset M} H^p(M, M\backslash K).$$

This is equal to $H_c^p(M)$ we defined at beginning. It is easy to see $H_c^p(M) \subset \varinjlim_{K\subset M} H^p(M, M\backslash K)$ by definition. For the other inclusion, each element of $\varinjlim_{K\subset M} H^p(M, M\backslash K)$ is represented by a cocycle in $S^p(M, M\backslash K)$ for some compact K, hence the inclusion at cochain level. And such a cocycle is zero in $\varinjlim_{K\subset M} H^p(M, M\backslash K)$ if and only if it is in $B^p(M, M\backslash L)$ for some compact $L \supset K$, hence the inclusion passes to the cohomology level.

Example 3.1.5. *We compute $H_c^*(\mathbb{R}^n)$. Since every compact set of $\mathbb{R}^n$ is contained in the closed ball $D_R(0)$ of some radius $R \in \mathbb{N}$, we have*

$$\varinjlim_{R\in\mathbb{N}} H^p(\mathbb{R}^n, \mathbb{R}^n\backslash D_R(0)) = \varinjlim_{K\subset\mathbb{R}^n} H^p(\mathbb{R}^n, \mathbb{R}^n\backslash K).$$

Now for any $R > 0$, $\mathbb{R}^n\backslash D_R(0)$ is homotopy equivalent to S^{n-1}. Hence the long exact sequence for pairs $(\mathbb{R}^n, \mathbb{R}^n\backslash D_R(0))$ gives us

$$H^m(\mathbb{R}^n, \mathbb{R}^n\backslash D_R(0)) = \begin{cases} \mathbb{Z} & m = n \\ 0 & m \neq n \end{cases}$$

Since the map $H^n(\mathbb{R}^n, \mathbb{R}^n\backslash D_R(0)) \to H^n(\mathbb{R}^n, \mathbb{R}^n\backslash D_{R+1}(0))$ corresponds to the inclusions are isomorphism, we conclude

$$H_c^m(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & m = n \\ 0 & m \neq n \end{cases}$$

This example tells us $H_c^*(M)$ is NOT a cohomology theory in the sense of Eilenberg-Steenrod. Since it is not a homotopy invariant: A point is compact, so $H_c^*(pt) = H^*(pt)$, but $H_c^*(\mathbb{R}^n)$ is not the same as $H^*(\mathbb{R}^n)$. Actually, it is even not a functor: the constant map $\mathbb{R}^n \to pt$ does not induce a map on cohomology with compact support.

3.2 Orientations for Manifolds

Suppose M is an n-manifold. For each $x \in M$, choose an open ball U with $x \in U$. Then by excision,

$$H_n(M, M\backslash x) = H_n(U, U\backslash x) = \mathbb{Z}.$$

Definition 3.2.1. *A local orientation μ_x for M at x is a choice of one of the two possible generators for $H_n(M, M\backslash x)$.*

If $x, y \in U$ then we have homomorphisms ρ_x and ρ_y induced by the inclusion of pairs

$$H_n(M, M\backslash x) \overset{\rho_x}{\leftarrow} H_n(M, M\backslash U) \overset{\rho_y}{\rightarrow} H_n(M, M\backslash y).$$

So a generator for $H_n(M, M\backslash U)$ gives a local orientation at any point in U.

Definition 3.2.2. *An orientation of M is a function $x \mapsto \mu_x$ subject to the following continuity condition: Given any point $x \in M$, there exists a neighborhood U of x and an element $\mu_U \in H_n(M, M\backslash U)$ such that $\rho_y(\mu_U) = \mu_y$ for each $y \in U$.*

We say M is *orientable* if there exists an orientation. And if the orientation is fixed, M is *oriented*.

One could translate this into the connectedness of the double cover

$$\tilde{M} = \{\mu_x | x \in M, \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

We will not go through this construction, see Hatcher for details. But we summarize two useful criterion

Proposition 3.2.3. *1. If M is simply connected, then M is orientable.*

2. Suppose $H^1(M; \mathbb{Z}_2) = 0$, then M is orientable.

Example 3.2.4. *1. $\mathbb{R}^n$, S^n, $\mathbb{C}P^n$ are orientable.*

2. If M and N are orientable, then $M \times N$ is orientable.

To determine which manifolds are not orientable, we have the following lemma. For the proof, we need relative Mayer-Vietoris sequence which is assigned in Example sheet 1:

$$\cdots \to H_n(X, A\cap B) \to H_n(X, A) \oplus H_n(X, B) \to H_n(X, A\cup B) \to H_{n-1}(X, A\cap B) \to \cdots$$

Lemma 3.2.5. *Let M be a n-manifold, and $K \subset M$ be a compact subset. Then*

1. $H_i(M, M\backslash K) = 0$ *for* $i > n$;
2. *Suppose* $x \mapsto a_x$ *is an orientation of* M. *Then there is a unique class* $a_K \in H_n(M, M\backslash K)$ *whose image in* $H_n(M, M\backslash x)$ *is* a_x *for all* $x \in K$.

Proof. We want to use the relative Mayer-Vietoris sequence for a triple (X, A, B):

$$\cdots \to H_p(X, A \cap B) \to H_p(X, A) \oplus H_p(X, B) \to H_p(X, A \cup B) \to \cdots$$

We write our proof in 4 steps:

1. Suppose the lemma is true for K_1, K_2 and $K_1 \cap K_2$, we want to prove it is true for $K_1 \cup K_2$ as well. Then we take $X = M$, $A = M\backslash K_1$, $B = M\backslash K_2$, so

$$\cdots \to H_{p+1}(M, M\backslash(K_1 \cap K_2)) \to H_p(M, M\backslash K) \to H_p(M, M\backslash K_1) \oplus H_p(M, M\backslash K_2) \to \cdots$$

By assumption, if $p > n$ the left and right term are both zero, so $H_p(M, M\backslash K) = 0$.

For the second statement, we know the map $H_n(M, M\backslash K_1) \oplus H_n(M, M\backslash K_2) \to H_n(M, M\backslash(K_1 \cap K_2))$ is the difference map $a_{K_1} - a_{K_2}$. By uniqueness, it has to be zero. So we have a_K from the exact sequence. This is unique because, $H_{n+1}(M, M\backslash(K_1 \cap K_2)) = 0$.

2. We reduce the problem to the case $M = \mathbb{R}^n$. Any compact set $K \subset M$ can be written as $K_1 \cup \cdots \cup K_m$ where each K_i is contained in a neighborhood which is homeomorphic to a ball in $\mathbb{R}^n$. Then apply step 1 and induction on $K_1 \cup \cdots \cup K_{m-1}$, K_m and their intersection.

3. Suppose $M = \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is a compact convex subset.

For any point $x \in K$, and S is a large $(n-1)$-sphere with centre x. Then S is a deformation retract of both $\mathbb{R}^n\backslash x$ and $\mathbb{R}^n\backslash K$. Hence the map

$$H_i(\mathbb{R}^n, \mathbb{R}^n\backslash K) \to H_i(\mathbb{R}^n, \mathbb{R}^n\backslash x)$$

is an isomorphism for each i.

Induction also shows this is true when K is a finite union of compact convex sets.

4. Now suppose $K \subset \mathbb{R}^n$ is an arbitrary compact subset and $\beta \in H_i(\mathbb{R}^n, \mathbb{R}^n\backslash K)$. We choose a relative cycle z with $[z] = \beta$. Let $C \in \mathbb{R}^n\backslash K$ be the union of the images of the boundary of singular simplices in z. C is compact (could be $\emptyset$ if z is closed in the absolute sense), the distance from K to C is some real number $\delta > 0$.

Cover K by finitely many balls with centres in K and radii $< \delta$. Let N be the union of these balls and so $K \subset N$ and z defines a class $\beta_N \in H_i(\mathbb{R}^n, \mathbb{R}^n\backslash N)$ so that the restriction $\rho_K(\beta_N) = \beta$.

If $i > n$ then by step 3, $\beta_N = 0$ so $\beta = 0$. This with step 2 finishes the first part of the lemma.

If $i = n$, step 1 and step 3 also construct a_N and then $a_K = \rho_K(a_N)$ such that $\rho_x(a_N) = a_x$ and $\rho_x(a_K) = a_x$. We prove the uniqueness: if a'_K is another choice, let $\beta = a_K - a'_K$. So $\rho_x(\beta) = 0$ for any $x \in K$, especially when x is one of the centres of the balls to define N. By step 3 again, β is zero on these balls and thus on N. Hence $a_K - a'_K = \beta = 0 \in H_n(\mathbb{R}^n, \mathbb{R}^n \backslash K)$. □

When M is closed (i.e. compact without boundary), take $K = M$, we have

Corollary 3.2.6. *Suppose M is a connected closed n-manifold. Then*

1. *$H_i(M) = 0$ if $i > n$;*
2. *M is orientable if and only if $H_n(M) = \mathbb{Z}$. If M is not orientable, $H_n(M) = 0$.*

Proof. Need a bit words for non-orientable case. If $H_n(M) \neq 0$, take a cycle $z \neq 0$. We take a cell decomposition of M. Then at two sides of any $n-1$-dimensional cell, the coefficient of z is the same. Since M is connected, the coefficient on all n-cells are the same. This gives us an orientation on M. □

Example 3.2.7. *$\mathbb{R}P^{2n}$ is not orientable, since $H_{2n}(\mathbb{R}P^{2n}) = 0$. $\mathbb{R}P^{2n+1}$ is orientable since $H_{2n+1}(\mathbb{R}P^{2n+1}) = \mathbb{Z}$.*

In particular, if M itself is compact, then there is one and only one $\mu_M \in H_n(M)$ with the required property. This class $\mu = \mu_M$ is called the *fundamental homology class* of M.

3.3 Poincaré duality theorem

The Poincaré duality for compact manifolds could be stated now.

Theorem 3.3.1. *Let M be a compact and oriented n-manifold, then the homomorphism*

$$D : H^p(M) \to H_{n-p}(M), \alpha \mapsto \alpha \cap \mu_M$$

is an isomorphism.

It actually follows from a more general theorem (which we will prove), for any oriented manifolds. Before stating the result, we need to explain the notations.

First observe that for any pair (X, A), the cap product gives rise to a paring

$$S^i(X, A) \otimes S_n(X, A) \to S_{n-i}(X)$$

and hence to pairing

$$H^i(X, A) \otimes H_n(X, A) \to H_{n-i}(X).$$

For oriented M, we define the duality map

$$D : H_c^p(M) \to H_{n-p}(M)$$

as follows. For any $a \in H_c^p(M) = \varinjlim H^p(M, M \backslash K)$, choose a representative $a' \in H^p(M, M \backslash K)$ and set

$$D(a) = a' \cap \mu_K.$$

This is well defined since for $K \subset L$, we have the restriction

$$\rho_K : H_n(M, M \backslash L) \to H_n(M, M \backslash K)$$

with $\rho_K(\mu_L) = \mu_K$. Then the naturality of the cap product tells us the following diagram commutes:

$$\begin{array}{ccc} H^i(M, M\backslash K) & & \\ & \searrow^{\cap \mu_K} & \\ \downarrow & & H_{n-i}(M) \\ & \nearrow_{\cap \mu_L} & \\ H^i(M, M\backslash L) & & \end{array}$$

Theorem 3.3.2. *Let M be an oriented n-manifold, then the homomorphism*

$$D : H_c^p(M) \to H_{n-p}(M)$$

is an isomorphism.

Proof. **1.** We first prove it for $M = \mathbb{R}^n$. Given a closed ball B, we know that $H_n(\mathbb{R}^n, \mathbb{R}^n \backslash B) = \mathbb{Z}$ with generator μ_B. Hence $H^n(\mathbb{R}^n, \mathbb{R}^n \backslash B) = \mathbb{Z}$ and by universal coefficient theorem, the homomorphism $h : H^n(\mathbb{R}^n, \mathbb{R}^n \backslash B; \mathbb{Z}) \to Hom(H_n(\mathbb{R}^n, \mathbb{R}^n \backslash B); \mathbb{Z})$ is an isomorphism. Then there exists a generator a such that $< a, \mu_B >= 1$. Now the identity

$$< 1 \cup a, \mu_B >=< 1, a \cap \mu_B >$$

shows that $a \cap \mu_B$ is a generator of $H_0(\mathbb{R}^n) = \mathbb{Z}$. Thus $\cap \mu_B$ gives an isomorphism $H^n(\mathbb{R}^n, \mathbb{R}^n \backslash B) \to H_0(\mathbb{R}^n)$ for all B. Hence by universal property of direct limit, the map D is an isomorphism in the case $i = n$. The cases $i \neq n$ is obvious since it maps 0 to 0.

2. Suppose $M = U \cup V$ and that the theorem is true for $U, V, U \cap V$. We first construct Mayer-Vietoris for H_c^*:

$$\cdots \to H_c^{p-1}(M) \to H_c^p(U \cap V) \to H_c^p(U) \oplus H_c^p(V) \to H_c^p(M) \to \cdots$$

This is obtained from relative Mayer-Vietoris

$$H^p(M, M\backslash(K\cap L)) \to H^p(M, M\backslash K)\oplus H^p(M, M\backslash L) \to H^p(M, M\backslash(K\cup L))$$

and excisions

$$H^p(M, M\backslash(K \cap L)) = H^p(U \cap V, U \cap V\backslash(K \cap L))$$
$$H^p(M, M\backslash K) = H^p(U, U\backslash K)$$
$$H^p(M, M\backslash L) = H^p(V, V\backslash L)$$

Now if we know the following diagram of exact sequence is commutative (up to a sign)

$$\begin{array}{ccccccc} H_c^p(U\cap V) & \longrightarrow & H_c^p(U)\oplus H_c^p(V) & \longrightarrow & H_c^p(M) & \xrightarrow{\delta} & H_c^{p+1}(U\cap V) \\ \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D \\ H_{n-p}(U\cap V) & \longrightarrow & H_{n-p}(U)\oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(M) & \xrightarrow{\partial} & H_{n-p-1}(U\cap V) \end{array}$$

then five lemma will finish this step.

The first two squares are easily seen to commute at the chain level. Much less simple is the third square, which we will show commutes up to a sign.

Notice we only need to show the following is commutative

$$\begin{array}{ccc} H^p(M, M\backslash(K\cup L)) & \xrightarrow{\delta} & H^{p+1}(U\cap V, U\cap V\backslash(K\cap L)) \\ \downarrow \cap\mu_{K\cup L} & & \downarrow \cap\mu_{K\cap L} \\ H_{n-p}(M) & \xrightarrow{\partial} & H_{n-p-1}(U\cap V) \end{array}$$

Let $A = M\backslash K$ and $B = M\backslash L$. Then the δ map is obtained from the short exact sequence

$$0 \to S^*(M, A)\cap S^*(M, B) \to S^*(M, A)\oplus S^*(M, B) \to S^*(M, A\cap B) \to 0$$

Recall that we use the fact $S^*(M, A\cup B) \to S^*(M, A)\cap S^*(M, B)$ induces isomorphism on cohomology. For a cocycle $\phi \in S^*(M, A\cap B)$, we write $\phi = \phi_A - \phi_B$ for $\phi_A \in S^*(M, A)$ and $\phi_B \in S^*(M, B)$. Then $\delta[\phi]$ is represented by the cocycle $\delta\phi_A = \delta\phi_B \in S^*(M, A)\cap S^*(M, B)$. Similarly if $z \in S_*(M)$ represents a homology class then $\partial[z] = [\partial z_U]$, where $z = z_U - z_V$ with $z_U \in S_*(U)$ and $z_V \in S_*(V)$.

Via barycentric subdivision, the class $\mu_{K\cup L}$ can be represented by a chain α that is a sum $\alpha = \alpha_{U\backslash L} + \alpha_{U\cap V} + \alpha_{V\backslash K}$ of chains in three open sets $U\backslash L$, $U\cap V$, and $V\backslash K$ respectively. By uniqueness of $\mu_{K\cap L}$ the chain $\alpha_{U\cap V}$ represents $\mu_{K\cap L}$, since the other two chains lie in the complement of $K\cap L$. Similarly the chain $\alpha_{U\backslash L} + \alpha_{U\cap V}$ represents μ_K.

Now let ϕ be a cocycle representing an element in $H^p(M, M\backslash(K \cup L))$. By δ, it maps to $\delta\phi_A$. Continuing downward to the bottom, we obtain $\delta\phi_A \cap \alpha_{U\cap V}$, which represents the same homology class as $(-1)^{n-p-1}\phi_A \cap \partial\alpha_{U\cap V}$, since

$$\partial(\phi_A \cap \alpha_{U\cap V}) = (-1)^{n-p}\delta\phi_A \cap \alpha_{U\cap V} + \phi_A \cap \partial\alpha_{U\cap V}.$$

For the other way, ϕ is first mapped to $\phi \cap \alpha \in H_{n-p}(M)$. Write is as a sum of a chain in U and a chain in V:

$$\phi \cap \alpha = \phi \cap \alpha_{U\backslash L} + \phi \cap (\alpha_{U\cap V} + \alpha_{V\backslash K}).$$

and by definition $\partial[\phi \cap \alpha] = [\partial(\phi \cap \alpha_{U\backslash L})] \in H_{n-k-1}(U \cap V)$.

$$\partial(\phi \cap \alpha_{U\backslash L}) = \phi \cap \partial\alpha_{U\backslash L} = \phi_A \cap \partial\alpha_{U\backslash L} = -\phi_A \cap \partial\alpha_{U\cap V}.$$

The second equality is because ϕ_B is zero on $M\backslash L$. The last equality follows from $\alpha_{U\backslash L} + \alpha_{U\cap V} = \mu_K$ which is a chain in $U\backslash K$. This completes step 2.

3. Suppose M is the union of a direct system of open subsets $\{U_i\}_{i\in I}$ with the property that if K is a compact subset of M then K is contained in some U_i. Then if we have

$$H_c^p(M) = \varinjlim_{i\in I} H_c^p(U_i), H_{n-p}(M) = \varinjlim_{i\in I} H_{n-p}(U_i),$$

and know the theorem is true for each U_i, then the theorem follows for M since the direct limit of a system of isomorphisms is an isomorphism.

We prove $H_p(M) = \varinjlim_{i\in I} H_p(U_i)$, the other follows by the same argument. By the universal property of directed limit, we have homomorphism

$$\varinjlim_{i\in I} H_p(U_i) \to H_p(M).$$

To show it is surjective: if $z \in S_p(M)$ is a cycle then there exists a compact set such that $[z] \in im(H_p(K) \to H_p(X))$. Assume $K \subset U_i$. Then $[z] \in im(H_p(U_i) \to H_p(M)$ and so $[z] \in im(\varinjlim_{i\in I} H_p(U_i) \to H_p(M))$.

For the injectivity, take a cycle z in U_i and assume it is a boundary of K in X. Then take j so that $K \subset U_j$ hence its inclusion into $\varinjlim_{i\in I} H_p(U_i)$ is 0.

4. Suppose M is an open subset of $\mathbb{R}^n$. If M is convex, it follows from step 1 since then M is homeomorphic to $\mathbb{R}^n$. We can find convex open sets $V_1, V_2, \cdots$ such that $M = \cup_{i=1}^\infty V_i$ (for example, those open discs whose centres have rational coordinates). Then by step 2, this is true for $V_1 \cup \cdots \cup V_r$ for each r. And by step 3, it is true for $\cup_r(\cup_{i=1}^r V_i) = \cup_{i=1}^\infty V_i = M$.

5. M is arbitrary. Consider the family of all open subsets U of M such that Poincaré duality holds for U. This family is nonempty. In view of step 3, we could apply Zorn's lemma to this family to choose a maximal open set V belonging to it. If $V \neq M$, then there is an open subset $B \subset M$ such that

B is homeomorphic to $\mathbb{R}^n$, and B is not contained in V. We apply step 2 and step 4 (for the intersection) to conclude Poincaré duality also holds for $V \cup B$, contradicting the maximality of V. Thus $V = M$. □

We also have Poncaré duality for non-orientable manifolds, but only for $\mathbb{Z}_2$ coefficient. Let M be an arbitrary n-manifold. For each point $x \in M$, μ_x denotes the unique non-zero element of the local homology group $H_n(M, M\backslash x; \mathbb{Z}_2)$. And for each compact subset K, the same argument of Lemma 3.2.5 gives us the unique element μ_K of $H_n(M, M\backslash K; \mathbb{Z}_2)$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$. Now we define homomorphism

$$H^p(M, M\backslash K; \mathbb{Z}_2) \to H_{n-p}(M; \mathbb{Z}_2), x \mapsto x \cap \mu_K.$$

This induces the homomorphism

$$D_2 : H^p_c(M; \mathbb{Z}_2) \to H_{n-p}(M; \mathbb{Z}_2).$$

Theorem 3.3.3. *For any n-manifold M, then the homomorphism*

$$D_2 : H^p_c(M; \mathbb{Z}_2) \to H_{n-p}(M; \mathbb{Z}_2)$$

is an isomorphism.

Chapter 4

Applications of Poincaré duality

4.1 Intersection form, Euler characteristic

4.1.1 Intersection pairing

Let M be a closed connected orientable n-manifold. We let $\mu \in H_n(M)$ be the orientation, i.e. the unique element such that the image of μ in $H_n(M, M\backslash x)$ is a generator.

Now we have a pairing on cohomology ring induced by cup product:

$$<,>: H^k(M) \times H^{n-k}(M) \xrightarrow{\cup} H^n(M) \xrightarrow{\cap\mu} \mathbb{Z}$$

In other words, $< a, b >= (a \cup b)(\mu)$. This map is called the *intersection forms.* One could also define the pairing on homology by taking Poincaré duality.

Poincaré duality simply tells us that the intersection form is non-singular when we take the free part.

Corollary 4.1.1. *Suppose we take coefficients in a field F. then the intersection form*

$$<,>: H^k(M; F) \times H^{n-k}(M; F) \to F$$

is non-singular. The same conclusion if we look at the the pairing

$$\frac{H^k(M;\mathbb{Z})}{Tors} \times \frac{H^{n-k}(M;\mathbb{Z})}{Tors} \to \mathbb{Z}.$$

Here the intersection form is nonsingular means both $< \alpha, \cdot >$ and $< \cdot, \alpha >$ are isomorphisms if α is non-zero.

Proof. Consider the composition

$$H^{n-k}(M; R) \xrightarrow{h} Hom_R(H_{n-k}(M; R), R) \xrightarrow{D^*} Hom_R(H^k(M; R), R)$$

Recall (from Example sheet 1) that h is an isomorphism for the above two cases. And here D^* is the Hom-dual of the Poincaré duality map $D : H^k \to H_{n-k}$. Notice

$$D^*(h(\alpha))(\beta) = \alpha \cap (\beta \cap \mu) = (\alpha \cup \beta)(\mu) =< \alpha, \beta > .$$

Since both D^* and h are isomorphisms, their composition is an isomorphism and hence the intersection form is non-singular. □

Later we will just write $H^m(M)$ for the free part if there is no confusion.

Corollary 4.1.2. *Let M be even dimensional ($n = 2m$) orientable manifold. Then the pairing*

$$H^m(M) \times H^m(M) \to \mathbb{Z}$$

is unimodular: choose a basis $u_1, \cdots , u_k \in H^m(M)$, then the matrix $A = (a_{ij})$ with $a_{ij} =< u_i, u_j >$ has $\det A = \pm 1$.

And when m is even, it is symmetric; m is odd, it is anti-symmetric.

Proof. The second conclusion is easy, only prove the first one. Take a basis $u_1, \cdots , u_k \in H^m(M)$, then we know there is a dual basis $v_1, \cdots , v_k$ such that $< u_i, v_j >= \delta_{ij}$. Let $A = (a_{ij})$ where $a_{ij} =< u_i, u_j >$. Let B is matrix of base change: $v_j = \sum_k b_{kj} u_k$. Then

$$\delta_{ij} =< u_i, v_j >= \sum_k b_{kj} < u_i, u_k >= \sum_k a_{ik} b_{kj},$$

i.e. $AB = I$. Since both of A, B are of $\mathbb{Z}$ coefficients, then $\det A = \pm 1$. □

Now, if an orientable M has dimension $4m$, the intersection pairing is a symmetric unimodular bilinear form. So the eigenvalues are all real numbers. We will denote the numbers of its positive and negative eigenvalues by b^+_{2m} and b^-_{2m} respectively. Their sum is Betti number b_{2m}. Their difference is an important invariant called *signature*, denote by

$$\sigma(M) = b^+_{2m} - b^-_{2m}.$$

For orientable manifolds of dimensions other than $4m$, let $\sigma(M) = 0$.

There is another viewpoint of the intersection pairing, from homology. If $x, y \in H_*(M)$ and $\xi, \eta \in H^*(M)$ and $x = D\xi, y = D\eta$. Then we define

$$x \cdot y =< \xi, \eta >= (\xi \cup \eta)(\mu_M)$$

which is also a non singular pairing by above corollary.

Assume X, Y are closed oriented submanifold of dimensions i and j respectively with $i + j = n$. We also assume they intersect transversally, i.e. at each point $x \in X \cap Y$,

$$T_x X + T_x Y = T_x M.$$

Then the intersection is also a submanifold of dimension 0, thus finite number of points. Then each $x \in X \cap Y$ has a sign $\epsilon(x)$ determined by comparing the orientations of $T_xX + T_xY$ and T_xM. Let $a = i_*(\mu_X)$ and $b = i_*(\mu_Y)$ where i is the inclusion. Then the intersection number could be calculated as

$$a \cdot b = \sum_{x \in X \cap Y} \epsilon(x). \tag{4.1}$$

The intuition is very clear. We choose a singular decomposition of manifold M such that it also induces singular decomposition of X and Y. So each intersection point will be a vertex of singular simplices. Since oriented means we have $\mu_M = \sum s_i$ where s_i are all simplices. And we also have similar formula for μ_X and μ_Y constitutes of sub-simplices of s_i, but might have a sign. Then $PD^{-1}(a) \cup PD^{-1}(b)(\sum s_i)$ is nonzero only when s_i contains intersection point. And at each intersection point, and any s_i containing it, the evaluation is just a check of whether the orientations are matched. To make this argument rigorous, we need to make the definition of the cohomology class $PD^{-1}(a)$ clearer such that it satisfies the property we want: restricting to each normal direction is just 1. This needs the Thom isomorphism theorem.

4.1.2 Betti numbers and Euler characteristic

Let us introduce Betti numbers and Euler characteristic. For a CW complex M with finitely many cells in each dimension, let

$$b_i = rank(H_i(M; \mathbb{Z})) = \dim H_i(M; \mathbb{Q}) = rank(H^i(M; \mathbb{Z})) = \dim H^i(M; \mathbb{Q})).$$

$b_i(M)$ is called the i-th Betti number. The *Euler characteristic* of M is

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i(M).$$

More generally, we have the Poincaré series $P_M(t) = \sum_{i=0}^{\infty} b_i(M)t^i$, and $\chi(M) = P_M(-1)$.

Give M a finite cell structure, let $\alpha_i(M)$ be the number of i-cells, then we have $\chi = \sum_i (-1)^i \alpha_i(M)$. If we let $Q_M(t) = \sum_i \alpha_i t^i(M)$, this actually follows from the following more general result

Theorem 4.1.3. *Let M be a finite CW complex. then*

$$Q_M(t) - P_M(t) = (1 + t)R(t).$$

Proof. First if we have a short exact sequence for Abelian groups

$$0 \to A \to B \to C \to 0$$

and A, B, C are finitely generated, then $rankB = rankA + rankC$.

Then we look at short exact sequences

$$0 \to Z_q \to C_q \to B_{q-1} \to 0$$
$$0 \to B_q \to Z_q \to H_q \to 0$$

We have

$$\sum_q (rankC_q)t^q = \sum_q (rankZ_q)t^q + t\sum_q (rankB_q)t^q,$$
$$\sum_q (rankZ_q)t^q = \sum_q (rankB_q)t^q + \sum_q (rankH_q)t^q.$$

Add them we have

$$Q_M(t) = P_M(t) + (1+t)\sum_q (rankB_q)t^q.$$

□

From this alternative definition of Euler characteristic, it is easy to see $\chi(M)$ has counting property:

Proposition 4.1.4. *1. If A and B are subcomplex of a finite CW complex M, $\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$.*

2. If $\tilde{M}$ is a k-sheeted covering of M, then $\chi(\tilde{M}) = k\chi(M)$.

Then we have the following restrictions on Euler characteristic of manifolds.

Proposition 4.1.5. *1. If M is an odd dimensional closed manifold then $\chi(M) = 0$.*

2. If M is an orientable $4k+2$-dimensional manifold, then $\chi(M)$ is even.

Proof. If M is orientable and odd dimension, then by Poincaré duality, $\chi(M) = 0$. If not orientable, we know its double covering $\tilde{M}$ we constructed before is orientable and $\chi(M) = \frac{1}{2}\chi(\tilde{M}) = 0$.

For the second statement, we know $b_i = b_{n-i}$ by Poincaré duality. So

$$\chi(M) \equiv b_{2k+1} \mod 2.$$

We want to prove b_{2k+1} is even. Choose a basis of $H^{2k+1}(M)$, and let A be the matrix of intersection form under this basis. Since $\alpha \cup \beta = -\beta \cup \alpha$ when both are $2k+1$ dimensional. So A is antisymmetric: $A = -A^T$. So

$$\det A = \det A^T = \det(-A) = (-1)^{b_{2k+1}} \det A.$$

Since the matrix is non-degenerate, so b_{2k+1} is even. □

A non-orientable manifold could violate the second statement. For example, $\chi(\mathbb{R}P^2) = 1$.

4.2 Calculation of cohomology rings

4.2.1 Cohomology ring of $\mathbb{C}P^n$

We have calculated that

$$H^m(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & m = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

We claim the cohomology ring

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{n+1}}$$

where α has degree 2, i.e $\alpha \in H^2(\mathbb{C}P^n; \mathbb{Z})$. Since the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ induces an isomorphism on H^i for $i \leq 2n-2$, by induction on n, $H^{2i}(\mathbb{C}P^n; \mathbb{Z})$ is generated by α^i for $i < n$. By the corollary, there is an integer m such that the product $\alpha \cup m\alpha^{n-1} = m\alpha^n = 1$. Hence $m = \pm 1$, and our conclusion follows. Notice $S^2 \times S^4$ has the same cohomology groups as $\mathbb{C}P^3$. But by Künneth formula,

$$H^*(S^2 \times S^4; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha, \beta]}{\alpha^2, \beta^2}.$$

This is not isomorphic to $H^*(\mathbb{C}P^3; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^4}$.

The same calculation works for Quaternionic projective space $\mathbb{H}P^n = \frac{\mathbb{H}^{n+1}\backslash 0}{\sim}$ where $x \sim y$ if there exists $\lambda \in \mathbb{H}\backslash 0$ such that $\lambda x = y$. We conclude that

$$H^*(\mathbb{H}P^n; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{n+1}}$$

where α has degree 4.

Let $\mathbb{O}$ be the Cayley numbers. They are non-associative. We can form $\mathbb{O}P^1 = S^8$ and $\mathbb{O}P^2$ and

$$H^*(\mathbb{O}P^2; \mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^3}.$$

4.2.2 Cohomology ring of $\mathbb{R}P^n$ and Borsum-Ulam

Recall that

$$H_m(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0 \text{ or } m = n = 2k+1 \\ \mathbb{Z}_2 & m \text{ odd }, 0 < m < n \\ 0 & \text{otherwise} \end{cases}$$

Hence by universal coefficient theorem, we know

$$H^m(\mathbb{R}P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

Using Theorem 3.3.3, and the same argument above for $\mathbb{C}P^n$, we conclude that

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[x]}{x^{n+1}}.$$

Now as an application of this calculation, we have

Lemma 4.2.1. *Suppose we have a continuous map $f : \mathbb{R}P^m \to \mathbb{R}P^n$ such that $f_* \neq 0 : H_1(\mathbb{R}P^m;\mathbb{Z}_2) \to H_1(\mathbb{R}P^n;\mathbb{Z}_2)$, then $m \leq n$.*

Proof. Since $H^1(X;G) = Hom(H_1(X),G)$, we know $f^* \neq 0 : H^1(\mathbb{R}P^n;\mathbb{Z}_2) \to H^1(\mathbb{R}P^m;\mathbb{Z}_2)$.

Take $\xi \neq 0 \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$, then $\eta = f^*(\xi) \neq 0 \in H^1(\mathbb{R}P^m;\mathbb{Z}_2)$. By the calculation of cohomology ring, we know $\eta^m = f^*(\xi^m) \neq 0$. So $\xi^m \neq 0 \in H^m(\mathbb{R}P^n;\mathbb{Z}_2)$, which means $m \leq n$. □

Lemma 4.2.2. *Let σ be a path connecting an antipodal on S^n. Then under the quotient map $\pi : S^n \to \mathbb{R}P^n$, it becomes a singular cycle $\pi_*(\sigma)$ representing a nonzero element in $H_1(\mathbb{R}P^n;\mathbb{Z}_2)$.*

Proof. We use the cellular decomposition induced from the natural one of $S^0 \subset S^1 \subset \cdots \subset S^n$. Let σ be the one connecting S^0. When $n = 1$, $\pi_*(\sigma)$ rotates along $\mathbb{R}P^1 = S^1$ odd number turns. It is nontrivial in $H_1(\mathbb{R}P^1;\mathbb{Z}_2)$.

When $n > 1$, take path τ with the same end points of σ in $S^1 \subset S^n$. By induction, $\pi_*(\tau)$ represents nonzero element in $H_1(\mathbb{R}P^1;\mathbb{Z}_2)$. Since the inclusion $H_1(\mathbb{R}P^1;\mathbb{Z}_2) \to H_1(\mathbb{R}P^n;\mathbb{Z}_2)$ is an isomorphism. So $\pi_*(\tau)$ is nonzero in $H_1(\mathbb{R}P^n;\mathbb{Z}_2)$ as well. On the other hand, $\sigma - \tau$ is singular cycle in S^n with $n > 1$, so it is a boundary. Hence $\pi_*(\sigma)$ is nonzero in $H_1(\mathbb{R}P^n;\mathbb{Z}_2)$. □

Theorem 4.2.3. *There is no continuous map $f : S^{n+1} \to S^n$ such that $f(-x) = -f(x)$.*

Proof. If there is such a map, then it gives a map $g : \mathbb{R}P^{n+1} \to \mathbb{R}P^n$. Take σ connecting antipodal of S^{n+1}, it is mapped to a path $f_*(\sigma)$ connecting the antipodal of S^n. Hence by Lemma 4.2.2, $g_* \neq 0 : H_1(\mathbb{R}P^{n+1};\mathbb{Z}_2) \to H_1(\mathbb{R}P^n;\mathbb{Z}_2)$. Then Lemma 4.2.1 finishes the proof. □

Corollary 4.2.4 (Borsuk-Ulam)**.** *Let $f : S^n \to R^n$ be a continuous map. Then there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Proof. If not, construct

$$g : S^n \to S^{n-1}, g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.$$

We have $g(-x) = -g(x)$, contradicting Theorem 4.2.3. □

Corollary 4.2.5 (Ham sandwich)**.** *Let $A_1, \cdots, A_m$ be m measurable sets in $\mathbb{R}^m$. Then we have a hyperplane P which bisects each A_i.*

Proof. Consider in $\mathbb{R}^{m+1}$. Fix $x_0 \notin \mathbb{R}^m$. For any vector $v \in S^m$, construct a hyperplane orthogonal to v and pass through x_0. It divides $\mathbb{R}^{m+1}$ and thus $\mathbb{R}^m$ into two parts. We record the volume of the $A_i \subset \mathbb{R}^m$ in the half space determined by the direction of v, by $f_i(v)$. Hence we have a continuous map

$$f : S^m \to \mathbb{R}^m, v \mapsto (f_1(v), \cdots, f_m(v)).$$

By corollary 4.2.4, there exists v wuch that $f(v) = f(-v)$. This hyperplane bisects each A_i. □

Another application is the Ljusternik-Schnirelmann category $cat(\mathbb{R}P^n) = n + 1$. This is because $cl(\mathbb{R}P^n) = n$, so $cat(\mathbb{R}P^n) \geq n + 1$. On the other hand, it is not hard to construct a smooth function on $\mathbb{R}P^n$ with $n + 1$ critical points (exercise).

4.2.3 Cohomology ring of lens spaces

Given an integer $m > 1$ and integers $l_1, \cdots, l_n$ relatively prime to m, define the *lens space* $L = L_m(l_1, \cdots, l_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ with the action of $\mathbb{Z}_m$ generated by

$$\rho(z_1, \cdots, z_n) = (e^{2\pi i l_1/m} z_1, \cdots, e^{2\pi i l_n/m} z_n).$$

The condition l_i is coprime to m means $\mathbb{Z}_m$ acts freely. Thus the projection $S^{2n-1} \to L$ is a covering space. When $m = 2$, ρ is the antipodal map and $L_2 = \mathbb{R}P^{2n-1}$.

L has a CW structure with one cell e^k for each $k \leq 2n - 1$ and the resulting cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Therefore

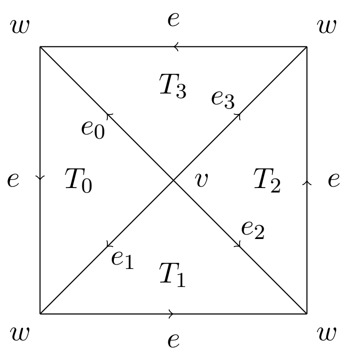
$$H_k(L_m(l_1, \cdots, l_n)) = \begin{cases} \mathbb{Z} & k = 0, 2n-1 \\ \mathbb{Z}_m & k \text{ odd }, 0 < k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

By universal coefficient theorem,

$$H^k(L_m(l_1, \cdots, l_n); \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & 0 \leq k \leq 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Since the cohomology group only depends on m and n, later on we will denote the lens space by L_m^{2n-1} or simply L^{2n-1}. To calculate the cup product, we let $\alpha \in H^1(L^{2n-1}; \mathbb{Z}_m)$ and $\beta \in H^2(L^{2n-1}; \mathbb{Z}_m)$ be generators.

We claim that $H^{2i}(L^{2n-1}; \mathbb{Z}_m)$ is generated by β^i and $H^{2i+1}(L^{2n-1}; \mathbb{Z}_m)$ is generated by $\alpha\beta^i$.



Use induction, we assume it is true for L^{2n-1} and want to show it for L^{2n+1}. Using the inclusion $L^{2n-1} \to L^{2n+1}$ which induces isomorphism in cohomology for $0 \leq k \leq 2n-1$ by comparing the cellular chain complexes, we may assume the claim holds for $H^k(L^{2n+1}; \mathbb{Z}_m)$ with $k \leq 2n-1$. By corollary 4.1.1, there exists $\lambda \in \mathbb{Z}_m$ such that $\beta \cup \lambda\alpha\beta^{n-1} = \lambda\alpha\beta^n$ generates $H^{2n+1}(L^{2n+1}; \mathbb{Z}_m)$. So λ has to a generator of $\mathbb{Z}_m$ and therefore $\alpha\beta^n$ is a generator of $H^{2n+1}(L^{2n+1}; \mathbb{Z}_m)$. It also implies that β^n is a generator of $H^{2n}(L^{2n}; \mathbb{Z}_m)$, otherwise $\alpha\beta^n$ would have order less than m.

To complete the calculation of the ring $H^*(L^{2n-1}; \mathbb{Z}_m)$, we need to compute α^2. By graded commutativity, we have $\alpha \cup \alpha = -\alpha \cup \alpha$. So if m is odd, $\alpha^2 = 0$.

When $m = 2k$, we claim $\alpha^2 = k\beta$. We use the fact that the 2-skeleton $S^1 \cup_{f_m} e^2$ of L^{2n-1} is the circle S^1 attached by a 2-cell with a map of degree m. We first get the 2-skeleton a Δ-complex structure by subdivide an m-gon into m triangles T_i around a central vertex v, and identify all the outer edges by rotations of the m-gon. We call the faces in a counterclockwise order $T_0, \cdots, T_{m-1}$ and the the rays from v which bound T_i by e_i and e_{i+1}. Then we choose a representative ϕ for α which assigns value 1 to the boundary edge. The condition ϕ is a cocycle means $\phi(e_i) + \phi(e) = \phi(e_{i+1})$, which means we could take $\phi(e_i) = i$ in $\mathbb{Z}_m$. Then by definition of the cup product, $(\phi \cup \phi)(T_i) = \phi(e_i)\phi(e) = i$. Since the sum $0 + 1 + \cdots + (m-1)$ is k in $\mathbb{Z}_m$, we know $\phi \cup \phi$ evaluates as k on $\sum T_i$. This means $\alpha^2 = k\beta$.

Hence

$$H^*(L^{2n-1}; \mathbb{Z}_{2k+1}) = \frac{\mathbb{Z}_{2k+1}[\alpha, \beta]}{\alpha^2 = 0, \beta^n = 0}$$

$$H^*(L^{2n-1}; \mathbb{Z}_{2k}) = \frac{\mathbb{Z}_{2k}[\alpha, \beta]}{\alpha^2 = k\beta, \beta^n = 0}$$

4.3 Degree and Hopf invariant

4.3.1 Degree

We can define degree of a map $f : M^n \to N^n$ between closed oriented connected manifolds. Indeed, by orientable manifolds $H_n(M) = H_n(N) = \mathbb{Z}$ and thus $f_* : H_n(M) \to H_n(N)$ maps the generator μ_M to an integer multiple k of μ_N. We call this $k := deg(f)$ the degree of the map f. The degree has natural composition property: $deg(f)deg(g) = deg(f \circ g)$. Since $H^n(M) = H^n(N) = \mathbb{Z}$ as well, we can define degree as the corresponding integer for the cohomology $f^* : H^n(N) \to H^n(M)$. Apparently, these two definitions result the same number.

Example 4.3.1. *A reflection of S^n along a great circle has degree -1, since it changes the orientation. Hence the antipodal map a sending $x \mapsto -x$ has degree $(-1)^{n+1}$ since it is a composition of $n+1$ reflections.*

This example has lots of corollaries. We only show a few.

Corollary 4.3.2. *1. If $f, g : S^n \to S^n$ are maps such that $f(x) \neq g(x)$ for all x then f is homotopic to $a \circ g$.*

2. If $f : S^n \to S^n$ has no fixed points then it is homotopic to the antipodal map, and thus has degree $(-1)^{n+1}$.

Proof. By assumption

$$x \mapsto \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

is a well defined homotopy from f to $a \circ g$.

The second follows from the first by taking $g = id$. □

We can also get some information for group actions on S^n. First note that S^{2n-1} could be viewed as the unit sphere in $\mathbb{C}^n$, thus it admits a free action of S^1, i.e. $z \mapsto e^{i\theta}z$. Especially, it tells us that $\mathbb{Z}_m$ could act on S^{2n-1} freely. But for S^{2n}

Corollary 4.3.3. *Suppose a group G acts freely on S^{2n}. Then $G \leq \mathbb{Z}_2$.*

Proof. By assumption, each non-trivial element $g \in G$ has no fixed point, thus has degree -1 by above corollary. Hence there is at most one such element, otherwise the composition would give a map of degree 1 which has to be trivial. □

Proposition 4.3.4. *Given $f : \mathbb{C}P^n \to \mathbb{C}P^n$, there exists an integer k such that* $\deg f = k^n$.

Proof. Let u be a generator of $H^2(\mathbb{C}P^n)$, then $f^*(u) = ku$ for some constant k. Hence $f^*(u^n) = f^*(u)^n = k^n u^n$. By definition, $\deg f = k^n$. □

Proposition 4.3.5. *If $f : S^{2n} \to \mathbb{C}P^n$ with $n > 1$ then $\deg(f) = 0$.*

Proof. $f^*(u) = 0$ since $H^2(S^{2n}) = 0$. So $f^*(u^n) = f^*(u)^n = 0$. □

Usually, the philosophy behind the last statement is: if there is a map $f : M \to N$ with $\deg(f) \neq 0$, then M should have more complicated topology then N.

Exercise:

1. Construct maps of any integer degrees for $f : S^n \to S^n$. (Hint: start with S^1 and then use suspension.)
2. Prove that any n-fold covering map has degree n.

4.3.2 Hopf invariant

Hopf invariant is a sort of degree when studying the maps $S^{2n-1} \to S^n$.

In general, given a map $f : S^m \to S^n$ with $m \geq n$, we can form the CW complex

$$C(f) := S^n \cup_f D^{m+1} = \frac{S^n \sqcup D^{m+1}}{f(x) \sim x, \forall x \in S^m}.$$

The homotopy type of $C(f)$ depends only on the homotopy class of f. We could use Proposition 1.7.9 to calculate the (co)homology group of it. For example, if $m = n$ and f has degree d. Then $H^n(C(f)) = \mathbb{Z}_{|d|}$, which detects degree up to sign.

When $m > n$, we calculate that the cohomology of $C(f)$ has $\mathbb{Z}$ in dimensions 0, n and $m+1$. Especially when $m = 2n-1$, we have chance to use cup product to detect something nontrivial. In this case, choose generators $\alpha \in H^n(C(f))$ and $\beta \in H^{2n}(C(f))$, then the ring structure of $H^*(C(f))$ is determined by $\alpha^2 = H(f)\beta$ for an integer $H(f)$ which is called the *Hopf invariant* of f.

If f is a constant map then $C(f) = S^n \vee S^{2n}$ and $H(f) = 0$. Also, $H(f)$ is always zero for odd n since $\alpha^2 = -\alpha^2$ in this case.

Example 4.3.6.

1. *$n = 2$. We use $S^2 = \mathbb{C}P^1$ and view S^3 as the unit sphere in $\mathbb{C}^2$. The map $S^3 \to S^2$ is defined as*

$$(z_0, z_1) \mapsto [z_0 : z_1].$$

From the definition, it is a bundle $S^1 \to S^3 \to S^2$, which is called the Hopf bundle. It is easy to see that $C(f) = \mathbb{C}P^2$. Thus $H(f) = 1$ since $H^*(\mathbb{C}P^2) = \frac{\mathbb{Z}[\alpha]}{\alpha^3}$.

2. *$n = 4$. Replacing the field $\mathbb{C}$ by $\mathbb{H}$, some construction yields the fiber bundle $S^3 \to S^7 \to S^4$. And $C(f) = \mathbb{H}P^2$ and $H(f) = 1$.*

3. *$n = 8$. Use Caylay octonion. We have Hopf bundle $S^7 \to S^{15} \to S^8$. And $C(f) = \mathbb{O}P^2$ and $H(f) = 1$.*

4. *$n = 1$. It is the covering map, viewed as bundle $S^0 \to S^1 \to \mathbb{R}P^1$. It is measured by its degree, which is 2.*

It is a fundamental theorem of Adams says that maps $S^{2n-1} \to S^n$ of Hopf invariant 1 only exists when $n = 2, 4, 8$. It has many interesting corollaries:

1. $\mathbb{R}^n$ is a division algebra only for $n = 1, 2, 4, 8$.

2. S^n has n linearly independent tangent vector fields only for $n = 0, 1, 3, 7$.

3. The only fiber bundle $S^p \to S^q \to S^r$ occur only when $(p, q, r) = (0, 1, 1), (1, 3, 2), (3, 7, 4), (7, 15, 8)$.

One could also define the Hopf invariant in terms of degree. Let y, z be two different regular values for a map $f : S^{2n-1} \to S^n$, then the manifolds $f^{-1}(y)$ and $f^{-1}(z)$ could be oriented and the linking number is defined as the degree of the function: Let M and N be two manifolds of dimension $n - 1$ in S^{2n-1}. Choose a point $p \in S^{2n-1}$ which is not in M or N, and think $S^{2n-1} \backslash p$ as $\mathbb{R}^{2n-1}$. Then the linking number $link(M, N)$ of M and N is defined as the degree of the map

$$g : M \times N \to S^{2n-2}, (x, y) \mapsto \frac{x - y}{||x - y||}.$$

Let us understand this definition in terms of low dimensional examples. First is the toy example: the linking of two S^0 in S^1 or $\mathbb{R}^1$. Let the coordinate of S^0 be $\{a, b\}$ and $\{c, d\}$ respectively. Then the map g is determined by the order of these numbers. For example, if $a < c < b < d$, then two S^0 are linked both from our common sense and from the formula since g maps to 1 once and -1 thrice, so degree one. If $a < b < c < d$, then degree is zero since maps to -1 four times. And if $a < c < d < b$, it maps to 1 and -1 both twice. But the degree is 0 as well since $g(a, c)$ and $g(a, d)$ is considered as opposite orientation because that of c and d.

A more realistic example is for two S^1 in S^3 or $\mathbb{R}^3$. So $g : T^2 \to S^2$. For a point $\mathbf{v}$ in the unit sphere, the orthogonal projection of the link to the plane perpendicular to $\mathbf{v}$ gives a link diagram on plane. A point in T^2 sends to $\mathbf{v}$ corresponds to a crossing in the link diagram where γ_1 is over γ_2. A neighborhood of it is mapped to a neighborhood of $\mathbf{v}$ preserving or reversing orientation depending on the sign of the crossing. Thus the it is just a signed counting of the number of times g covers $\mathbf{v}$.

There is a more concrete formula (Gauss formula) which could be generalized to higher dimension

$$link(\gamma_1, \gamma_2) = \frac{1}{4\pi}\int_0^{T_1}\int_0^{T_2} \frac{(\dot{\gamma_1}, \dot{\gamma_2}, \gamma_1 - \gamma_2)}{||\gamma_1 - \gamma_2||^3} dt_1 dt_2.$$

It is nothing but an integration interpretation of degree.

Now, the Hopf invariant $H(f) = link(f^{-1}(y), f^{-1}(z))$ for any two regular values y and z. To understand the equivalence of these definitions, we understand the cup product as the intersection of the Poincaré dual of cocycles. For the toy model above, S^1 bounds a D^2. Pairs of two points in S^1 are linked if and only if two semi circle in D^2 with the pairs as end points intersect. When glue the boundary S^1 by a double covering to another S^1, we get $\mathbb{R}P^2$, and two semi-circle above become two S^1.

For two S^1's, which are considered as inverse image of a regular value of map $S^3 \to S^2$, we have similar story. But now S^3 bounds D^4 where one could consider the picture in $\mathbb{C}^2$, and S^1 bounds an immersed disk. Then the intersection number of these two surfaces in D^4 is exactly the same as the linking number of two S^1 in S^3. One could prove this fact by pulling two circles until they touch. So the intersection is at the boundary and easy to look at. Finally, since S^1 are fibers of the map, after gluing they will become a point and the original surfaces will become a closed one.

For an explicit example bearing in mind, one could consider the Hopf bundle $S^1 \to S^3 \to S^2$ where the fiber is (z_1, z_2) with a fixed ratio. For example, $S^1 \times 0, 0 \times S^1 \subset S^3 \subset \mathbb{C} \times \mathbb{C}$ are two of them.

4.4 Alexander duality

Let us first introduce notations. The reduced (co)homology $\tilde{H}_i(M) = H_i(M, x)$ and $\tilde{H}^i(M) = H^i(M, x)$. So it only differs from original (co)homology at $i = 0$.

Theorem 4.4.1. *If K is a compact, locally contractible, nonempty, proper subspace of S^n, then $\tilde{H}_i(S^n \backslash K; \mathbb{Z}) = \tilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.*

Proof. We first argue it for $i \neq 0$. By Poincaré duality, $H_i(S^n \backslash K; \mathbb{Z}) = H_c^{n-i}(S^n \backslash K)$. By definition of cohomology with compact supports,

$$H_c^{n-i}(S^n \backslash K) = \varinjlim H^{n-i}(S^n \backslash K, U \backslash K)$$

where U is taken as open neighborhoods of K. By excision, $H^{n-i}(S^n \backslash K, U \backslash K) = H^{n-i}(S^n, U)$. And by long exact sequence for pairs, $H^{n-i}(S^n, U) = \tilde{H}^{n-i-1}(U)$ when $i \neq 0$. Now if we can show

$$\varinjlim \tilde{H}^{n-i-1}(U) = \tilde{H}^{n-i-1}(K),$$

the argument for $i \neq 0$ is complete.

To show this, we use the fact that K is a retract of some neighborhood U_0 in S^n since it is locally contractible. Thus in the directed limit we can only choose these open neighborhood $U \subset U_0$ which could be retracted to K. This implies the surjectivity of the map $\varinjlim H^*(U) \to H^*(K)$ since we can pull back the cohomology of K to that of U. To prove the injectivity, any $U \subset U_0$ is regarded as a subspace of $\mathbb{R}^n \subset S^n$. The linear homotopy $U \times I \to \mathbb{R}^n$ from the identity to the retraction $U \to K$ takes $K \times I$ to K, hence takes $V \times I$ to U for some (small) neighborhood V of K by compactness of I. Hence the inclusion $V \hookrightarrow U$ is homotopic to the retraction $V \to K \subset U$. Thus the restriction $H^*(U) \to H^*(V)$ factors through $H^*(K)$. Therefore if an element of $H^*(U)$ restrict to 0 in $H^*(K)$, it will be zero in $H^*(V)$ and thus in $\varinjlim H^*(U)$.

The only difference for $i = 0$ case is we do not have $H^n(S^n, U) = \tilde{H}^{n-1}(U)$. Instead we have the short exact sequence

$$0 \to \tilde{H}^{n-1}(U) \to H^n(S^n, U) \to H^n(S^n) \to 0$$

Take directed limit, we have seen the first term becomes $\varinjlim \tilde{H}^{n-1}(U) = \tilde{H}^{n-1}(K)$. By Poincaré duality, the middle term is $H_0(S^n \backslash K)$ and the last is $H_0(S^n) = \mathbb{Z}$. So this sequence tells us $\tilde{H}_0(S^n \backslash K) = \tilde{H}^{n-1}(K)$. □

In the proof, we know that the locally contractible condition is used to guarantee $\varinjlim \tilde{H}^{n-i-1}(U) = \tilde{H}^{n-i-1}(K)$. This is not always true. Look at the the following example.

Example 4.4.2. *Let K denote the subset of the graph of the function $y = \sin(\frac{1}{x})$ for $x \neq 0$ and y-axis with $|x|, |y| \leq 1$. Since there are three path components, $H^0(K; \mathbb{Z})$ is free abelian of rank 3.*

However, for the directed limit $\varinjlim H^0(U)$, we only need calculate it for open path connected neighborhoods of K, and thus $\varinjlim H^0(U) = \mathbb{Z}$.

Notice that K is not locally contractible at origin. Hence, it is also not a CW complex. This is called topologist's sine curve which is a typical example of a connected but not path connected space. We also notice that this fails the Alexander duality.

This theorem has many interesting applications. Let us start with the lowest nontrivial dimension $n = 2$.

Corollary 4.4.3. *Let $K \subset S^2$ be a simple closed curve, then $S^2 \backslash K$ has two components.*

Proof. Alexander duality says that $\tilde{H}_0(S^2 \backslash K) = H^1(S^1) = \mathbb{Z}$. So $H_0(S^2 \backslash K) = \mathbb{Z}^2$. □

For $n = 3$, if we take K as knots. The Alexander duality simply tells us that we cannot distinguish different knots from their homology groups. A result of Gordon-Luecke tells us that the fundamental group of the knot complement determines the knot. Actually, if we choose $K \subset S^n$ as a space homeomorphic to S^m, we will have the Alexander duality as well and the proof is a delightful use of Mayer-Vietoris sequence. Especially, this works for a Alexander horned sphere. It is an example homeomorphic to D^3 with its boundary homeomorphic to S^2, but its complement is not simply connected. However, by Alexander duality, its complement has trivial first homology.

Exercise: A non-orientable closed surface cannot be embedded in S^3 as a submanifold.

Finally, we want to add that if we work on Cech cohomology instead of singular cohomology, then the locally contractible condition could be removed. This is because Cech cohomology $\check{H}^q(K) = \varinjlim H^q(U)$ for all neighborhoods of K. Especially, for Cech cohomology H^0 detects the connected components instead of path connected components. Notice that it does not contradict to the Eilenberg-Steenrod uniqueness since all CW complexes are locally contractible, thus path connected if it is connected.

A final remark on the definition of Cech cohomology. It is defined first to each open cover $\mathcal{U} = \{U_\alpha\}$ of a given space X and associate a simplicial complex $N(\mathcal{U})$ called its nerve. This associates a vertex to each U_α and a set of $k+1$ vertices spans a k-simplex if the corresponding U_α have nonempty intersection. For a refinement $\mathcal{V} = \{V_\beta\}$ of $\mathcal{U}$, which means each V_β is contained in some U_α, then these inclusion would induce a simplicial map $N(\mathcal{V}) \to N(\mathcal{U})$. Then the Cech cohomology $\check{H}^q(X)$ is defined as $\varinjlim H^q(N(\mathcal{U}))$.

4.5 Manifolds with boundary

An n-manifold with boundary is a Hausdorff space M in which each point has an open neighborhood homeomorphic either to $\mathbb{R}^n$ (called *interior point*) or to the half space $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$ (called *boundary point*). An interior point $x \in M$ has $H_n(M, M\backslash x) = \mathbb{Z}$. A boundary point x corresponds to a point $(x_1, \cdots, x_n) \in \mathbb{R}^n_+$ with $x_n = 0$. By excision, we have $H_n(M, M\backslash x) = H_n(\mathbb{R}_n^+, \mathbb{R}_n^+\backslash 0) = 0$.

If M is a compact manifold with boundary, then ∂M has a collar neighborhood, i.e. an open neighoborhood homeomorphic to $\partial M \times [0, 1)$ by a homeomorphism sending ∂M to $\partial M \times 0$.

A compact manifold M with boundary is called orientable if $\mathring{M} := M\backslash\partial M$ is orientable. If $\partial M \times [0, 1)$ is a collar neighborhood of ∂M, then $H_i(M, \partial M) = H_i(M\backslash\partial M, \partial M \times (0, \frac{1}{2}))$. So Lemma 3.2.5 gives a relative fundamental class, denoted as $[M, \partial M]$ restricting to a given orientation at each point of $M \backslash \partial M$. The following tells how to relate relative fundamental class to $\mu_{\partial M}$. Later, for simplicity, we will write it as $[\partial M]$.

Proposition 4.5.1. *An orientation of M determines an orientation of ∂M.*

Proof. Consider an open neighborhood U of a point $x \in \partial M$ which is homeomorphic to an open half disk in $\mathbb{R}^n_+$. Let $V = \partial U = U \cap \partial M$ and let $y \in \mathring{U} = U \backslash V$. We have the following isomorphisms

$$\begin{aligned} H_n(\mathring{M}, \mathring{M} \backslash \mathring{U}) =& H_n(\mathring{M}, \mathring{M} \backslash y) \\ =& H_n(M, M \backslash y) \\ =& H_n(M, M \backslash \mathring{U}) \\ \xrightarrow{\partial}& H_{n-1}(M \backslash \mathring{U}, M \backslash U) \\ =& H_{n-1}(M \backslash \mathring{U}, (M \backslash \mathring{U}) \backslash x) \\ =& H_{n-1}(\partial M, \partial M \backslash x) \\ =& H_{n-1}(\partial M, \partial M \backslash V) \end{aligned}$$

The connecting homomorphism is that of the triple $(M, M \backslash \mathring{U}, M \backslash U)$ that is an isomorphism since $H_*(M, M \backslash U) = H_*(M, M) = 0$. The isomorphism that follows comes from the observation that the inclusion $(M \backslash \mathring{U}) \backslash x \to M \backslash U$ is a homotopy equivalence. The next to last isomorphism is given by excision of $\mathring{M} \backslash \mathring{U}$. $\square$

Especially, $\partial[M, \partial M]$ restricts to a generator of $H_{n-1}(\partial M, \partial M \backslash x)$ for all $x \in \partial M$ and thus is the fundamental class $[\partial M]$ determined by the orientation of ∂M which is induced from that of M.

We have the following

Theorem 4.5.2 (Lefschetz duality)**.** *Suppose M is a compact oriented n-manifold with boundary, then the homomorphisms*

$$D : H^p(M) \to H_{n-p}(M, \partial M), \alpha \mapsto \alpha \cap [M, \partial M]$$

$$D : H^p(M, \partial M) \to H_{n-p}(M), \alpha \mapsto \alpha \cap [M, \partial M]$$

are isomorphisms.

And the following diagram is commutative.

$$\begin{array}{ccccccc} H^{q-1}(M) & \longrightarrow & H^{q-1}(\partial M) & \xrightarrow{\delta} & H^q(M, \partial M) & \longrightarrow & H^q(M) \\ \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D \\ H_{n-q-1}(M, \partial M) & \xrightarrow{\partial} & H_{n-q}(\partial M) & \longrightarrow & H_{n-q}(M) & \longrightarrow & H_{n-q}(M, \partial M) \end{array}$$

Proof. We just apply Theorem 3.3.2 to $M \backslash \partial M$. Via a collar neighborhood, we have $H^p(M, \partial M) = H_c^p(M \backslash \partial M)$. And obviously, $H_{n-p}(M) = H_{n-p}(M \backslash \partial M)$. Hence $D : H^p(M, \partial M) \to H_{n-p}(M)$ is an isomorphism.

The commutativity could be checked by inspecting the definition and using the boundary formula for cap product. Leave the details as an exercise.

Finally by five lemma, $D : H^p(M) \to H_{n-p}(M, \partial M)$ is an isomorphism as well. □

For general manifold with boundary, we also similarly have $H_c^p(M)$ isomorphic to $H_{n-p}(M, \partial M)$, and $H_c^p(M, \partial M)$ isomorphic to $H_{n-p}(M)$ if we define $H_c^p(M, \partial M) = \varinjlim H^i(M, (M - K) \cup \partial M)$.

Now, we want to know what kind of n-manifolds could be the boundary of an $n+1$-dimensional manifold with boundary? From the classification of surfaces, we now each orientable closed surface is the boundary of certain 3-manifold. What about the non-orientable ones?

Theorem 4.5.3. *Let an n-manifold $M^n = \partial W^{n+1}$. Then $\chi(M)$ is even.*

Proof. When n is odd, it follows from Corollary 4.1.5.

When n is even, take the double of W, which is obtained by take two copies W^+ and W^- of W and glue them along the boundary. We denote it by $2W$. So $W^+ \cup W^- = 2W$ and $W^+ \cap W^- = M$. Hence, we have

$$\chi(2W) + \chi(M) = \chi(W^+) + \chi(W^-) = 2\chi(W).$$

Since $2W$ is an odd dimensional manifold, $\chi(2W) = 0$. Hence $\chi(M) = 2\chi(W)$, which is an even number. □

Example 4.5.4. *The double of a Möbius band is a Klein bottle. The double of an annulus of dimension 2 is a torus. The double of a disk is a sphere. Whose double is a torus?*

For non-orientable surface, $\mathbb{R}P^2$ is not a boundary. But one can check that the Klein bottle is a boundary. More generally, one can check that $\mathbb{R}P^{2k}$ and $\mathbb{C}P^{2k}$ are not boundaries.

The next result relates the signature with the boundaries.

Theorem 4.5.5. *Let $M^{4k} = \partial W^{4k+1}$ where W is a compact oriented $4k+1$-manifold, then $\sigma(M) = 0$.*

Proof. We use $\mathbb{R}$ as coefficient for (co)homology. We denote $[M] = \mu_M$.

The inclusion $i : M \to W$ induces homomorphism $i^* : H^{2k}(W) \to H^{2k}(M)$. Let $U = \text{Im } i^*$.

For $u = i^*(w) \in U$, we have

$$< u, u > = < i^*(w) \cup i^*(w), [M] > = < i^*(w \cup w), [M] > = < w \cup w, i_*[M] > = 0$$

The last equality is because $i_* \partial = 0$ in the long exact sequence of the pair (W, M) and $[M] = \partial [W, M]$.

We have seen that the diagram

$$\begin{array}{ccccc} H^{2k}(W) & \xrightarrow{i^*} & H^{2k}(M) & \xrightarrow{\delta} & H^{2k+1}(W,M) \\ \downarrow D & & \downarrow D & & \downarrow D \\ H_{2k+1}(W,M) & \xrightarrow{\partial} & H_{2k}(M) & \xrightarrow{i_*} & H_{2k}(W) \end{array}$$

is commutative. So $rkImi^* = rk\ker\delta = rk\ker i_*$. Since i^* and i_* are dual homomorphisms (that is where we use $\mathbb{R}$ coefficient), so rk coker$i^* = rk\ker i_*$. Hence

$$rkH^{2k}(M) = rkImi^* + rk \text{ coker} i^* = 2rkU.$$

Let $V = H^{2k}(M)$ and the positive/negative eigenspace of the intersection form $< v, v >$ would decompose it as $V^{\pm}$. The intersection form is 0 on linear subspace U, so $V^+ \cap U = 0$. Hence $rkV^+ + rkU \leq rkV$. Similarly $rkV^- + rkU \leq rkV$. However, the intersection is nonsingular, so $rkV^+ + rkV^- = rkV$.Thus $rkV^{\pm} = rkU$ and $\sigma(M) = 0$. □

This is proved by Thom, as part of his cobordism theory.

4.5.1 Linking number, Massey product

We could interpret the linking number of two cycles in Euclidean space by cup product in the complementary space. For example, suppose that S^p and S^q are disjoint spheres in S^n where $n = p+q+1$, and $1 \leq p < q \leq n-2$. By Alexander duality theorem, the complementary space $S^n \backslash (S^p \cup S^q)$ has cohomology group $\mathbb{Z}$ in dimensions p, q and $p+q$. Therefore the cup product of the generators in dimensions p and q will be a certain multiple of the generator of cohomology class in dimension $p+q$. It can be shown that this multiple is just the linking number of S^p and S^q. Another slightly different but easier to generalize definition is the following: we consider open neighoborhood of the linked spheres and let them be U_1, U_2, and M is the complement of U_1, U_2 in S^n and let the boundary be $B = B_1 \cup B_2$. Let w and v be the generators of H^p and H^q respectively. And let the generators $\mu \in H^n(M, B), \mu_i \in H^{n-1}(B_i)$. These could be chosen to the ones compatible with the orientations induced from S^n. Then there is a inclusion $g : B \to M$. We have the exact sequence

$$H^{n-1}(M) \xrightarrow{g^*} H^{n-1}(B) \xrightarrow{\delta} H^n(M, B) \to 0$$

Since μ_i are mapped to μ, the linking number is just the number m such that $g^*(w \cup v) = m(\mu_1 - \mu_2)$.

To understand it by intersections for two S^1 in S^3. Then the w, v corresponds to singular discs D_1, D_2 bounds the two S^1. And the generator of $H^2(S^3 \backslash (S^1 \cup S^1))$ corresponds to a path connecting B_1 and B_2. Hence the intersection, if in general position, is a signed count of these paths. Can you see the second viewpoint from this interpretation?

Similar idea could be applied to understand the Massey product. Now let us have three spheres in S^n, any two of them have linking number 0. The generators of cohomology in dimensions between 0 and $n-1$ are denoted by w_1, w_2, w_3. We could similarly define μ_i and the same exact sequence. Notice g^* is injective since the 3 cohomology groups are FREE abelian groups of dimensions $2, 3, 1$. Another view to see this is by the naturality of Alexander duality, i.e. the following diagram commutes:

$$\begin{array}{ccc} H^q(X) & \longrightarrow & H^q(Y) \\ \downarrow D & & \downarrow D \\ H_{n-q-1}(S^n\backslash X) & \longrightarrow & H_{n-q-1}(S^n\backslash Y) \end{array}$$

Now triple product could be understood as higher linking number of links, like the ones for Borromean ring.

Theorem 4.5.6. *There exists an integer m_{13} such that*

$$g^* < w_1, w_2, w_3 >= m_{13}(\mu_1 - \mu_3).$$

For Borromean ring, this integer is 1. Here is a sketch of the proof. If $x \in H^{n-1}(M)$ and $g^*(x) = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$, it follows from the exactness that $a_1 + a_2 + a_3 = 0$. Hence we only need to prove the coefficient μ_2 in $g^* < w_1, w_2, w_3 >$ is 0, or $g_2^* < w_1, w_2, w_3 >= 0$ for $g_2 : B_2 \to M$. Actually, we could show $g_2^* w_1 = g_2^* w_3 = 0$, which follows from the naturality of Alexander duality and the fact that pairwise linking numbers are 0.I.e. understand g_2^* on q_1th cohomology as

$$H_{p_1}(U_1) \oplus H_{p_1}(U_2) \oplus H_{p_1}(U_3) \to H_{p_1}(S^n\backslash B_2) = H_{p_1}(U_2) \oplus H_{p_1}(S^n\backslash \bar{U}_2)$$

Then w_1 corresponds to the generator of $H_{p_1}(U_1)$ where $U_1 \subset S^n\backslash \bar{U}_2$ and the degree of

$$H_{p_1}(U_1) \to H_{p_1}(S^n\backslash \bar{U}_2)$$

is just the linking number of S_1 and S_2.

Exercise: Think about how to use the intersection viewpoint to understand the m_{13} of Borromean ring.

4.6 Thom isomorphism

Let B be a manifold. A *vector bundle* $\pi : E \to B$ of rank n is a family of n-dimensional real vector spaces $\{E_x\}_{x\in B}$, with $E := \sqcup_{x\in M} E_x$ and π :

$E \to B$ mapping E_x to x, equipped with a topology for E such that π is continuous and the following local triviality condition holds:

For each $x \in B$ there exists a neighborhood U of x and a homeomorphism

$$t : E|_U := \pi^{-1}(U) \to U \times \mathbb{R}^n$$

which is fibre preserving in the sense that for all $x \in B$ the restriction of t on E_y is a vector space isomorphism onto $\mathbb{R}^n$.

The space E is called the *total space*, B is called the *base space*. The map π is called the *projection*. A continuous map $s : B \to E$ such that $\pi s = 1$ is called a *section* of E. We could view B as a subset of E via the zero section $x \mapsto 0 \in E_x$. Then denote $E_0 = E \backslash B$.

Typical examples include the trivial bundle $B \times \mathbb{R}^k$; the tangent bundle $T_M = \sqcup_{x\in M} T_x M$. Let $\{(U_i, \phi_i)\}$ be an atlas for M. The diffeomorphism $\phi_i : U_i \to \mathbb{R}^n$ induces a map

$$(\phi_i)_* : T_{U_i} \to T_{\mathbb{R}^n} = \mathbb{R}^n \times \mathbb{R}^n$$

and the normal bundle $N = N_{S/M}$ of S in M which is determined by the exact sequence

$$0 \to T_S \to T_M|_S \to N \to 0.$$

In the previous statements, we make use of a fact which is a good exercise.

Exercise: A vector bundle of rank n is trivial if and only if it has n sections which are linearly independent on $\pi^{-1}(x)$ for $\forall x \in B$.

A section of TM is a vector field on M. Especially, it implies that $T\mathbb{R}^n$ is trivial but TS^2 is not trivial.

Similar to that of a manifold, a local orientation at $x \in B$ is a preferred generator $\mu_x \in H^n(E_x, E_x \backslash 0)$. A vector bundle is called orientable if for every point $x \in B$, there is a neighborhood $x \in U \subset B$ such that the there is a cohomology class $\mu_U \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0)$ such that $\mu_U|_{E_x} = \mu_x$.

Theorem 4.6.1. *Let $\pi : E \to B$ be an oriented vector bundle of rank n. Then*

1. *$H^m(E, E_0) = 0$ for $m < n$.*
2. *There exists a unique cohomology class $u \in H^n(E, E_0)$, called the Thom class such that for all $x \in B$, the restriction of u to $H^n(E_x, E_x \backslash 0) = \mathbb{Z}$ is the prefered generator determined by the orientation.*
3. *The map*

$$T : H^m(B) \to H^{m+n}(E, E_0)$$
$$\alpha \mapsto \pi^* \alpha \cup u$$

is an isomorphism.

Proof. Let us only prove the case when B is compact. Then we can choose finite covering $\{U_i\}$ such that on each U_i, E is a trivial bundle.

1. First consider the case of trivial bundle $E = B \times \mathbb{R}^n$. Then by Künneth formula. We have

$$H^*(B) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \backslash 0) = H^*(E, E_0),$$

and hence

$$H^m(E, E_0) = H^{m-n}(B) \otimes \mathbb{Z} = H^{m-n}(B).$$

So $H^n(E, E_0) = \mathbb{Z}$, and we choose u as the generator corresponding to the orientation. Then the theorem is verified in this case.

2. We construct u by induction. Suppose $B = V \cup W$ where the where the assertions of the theorem hold for $E|_V, E|_W, E|_{V\cap W}$. Considering the long exact sequence of the pair (E, E_0) we have

$$H^{m-1}(E|_{V\cap W}, E_0|_{V\cap W}) \to H^m(E, E_0) \to H^m(E|_V, E_0|_V) \oplus H^m(E|_W, E_0|_W)$$

The the assertion 1 follows from assumption on V, W. For $m = n$, we have

$$0 \to H^n(E, E_0) \to H^n(E|_V, E_0|_V) \oplus H^n(E|_w, E_0|_W) \to H^n(E|_{V\cap W}, E_0|_{V\cap W}) \to \cdots$$

By assumption, the Thom classes u_V and u_W exist and are unique. By uniqueness, they have the same image in $E_{V\cap W}$, namely $u_{V\cap W}$. Thus they form a cohomology class $u \in H^n(E, E_0)$. This class is uniquely defined since $H^{n-1}(E, E_0) = 0$.

To show the last assertion, we consider the following diagram

$$\begin{array}{ccccc}
H^m(V) \oplus H^m(W) & \longrightarrow & H^m(V \cap W) & \xrightarrow{\delta} & H^{m+1}(B) \\
\downarrow T & & \downarrow T & & \downarrow T \\
H^{m+n}(E|_V, E_0|_V) \oplus H^{m+n}(E|_W, E_0|_W) & \longrightarrow & H^{m+n}(E|_{V\cap W}, E_0|_{V\cap W}) & \xrightarrow{\delta} & H^{m+n+1}(E, E_0)
\end{array}$$

If we could show it commutes then the 5-lemma would give the right T is also isomorphism. Again, the point is to show the second square commutes. Let choose a representative $\phi \in S^{m+n}(E, E_0)$ of u. Then the restrictions ϕ_V, ϕ_W and $\phi_{V\cap W}$ represent the Thom classes u_V, u_W and $u_{V\cap W}$ respectively. Now take $a \in H^k(V \cap W)$ and a representative ψ. Suppose $\delta a = b$ and if we write $\psi = \psi_V - \psi_W$ where $\psi_V \in S^k(V)$ and $\psi_W \in S^k(W)$, we have $[\delta\psi_V] = b$. Hence

$$T\delta(a) = \pi^*(b) \cup u = \pi^*[\delta\psi_V] \cup u.$$

Next

$$\delta T(a) = \delta(\pi^*(a) \cup u_{V\cap W}) = [\delta\pi^*(\psi_V) \cup \phi_V] = T\delta.$$

Second equality is because ϕ_V is closed, the last is because π^* commutes with δ since it is a cochain map.

3. Suppose B is covered by finitely many open sets $B_1, \cdots, B_k$ such that the bundle E_{B_i} is trivial for each B_i. We will prove by induction on k that the theorem is true for E. It is trivial for $k = 1$. For $k > 1$, we can assume the assertions are true for $E|_{B_1 \cup \cdots \cup B_{k-1}}$ and $E|_{(B_1 \cup \cdots \cup B_{k-1}) \cap B_k}$. Hence by step 2, the theorem follows. □

Now let M^n be a closed smooth manifold and S is a codimension k closed submanifold which is cooriented, i.e. the normal bundle N_S is an oriented vector bundle. So a natural coorientation would be induced from orientations of TS and TM. The tubular neighborhood theorem states that every submanifold S in M has a tubular neighborhood which is diffeomorphic to the normal bundle. Then we could indentify such a tubular neighborhood with our normal bundle $N_{S/M}$. And the Thom isomorphism applies to the normal bundle gives

$$H^*(S) \xrightarrow{T} H^{*+k}(N_S, N_S \backslash S) \to H^{*+k}(M, M \backslash S) \to H^{*+k}(M).$$

Without confusion, we denote the image of $1 \in H^0(S)$ in this sequence by Φ as the image of $u \in H^k(M, M\backslash S)$ in $H^k(M)$, and called the the Thom class of S. Actually, this is the inverse of Poincaré with respect to S, i.e

Proposition 4.6.2. $i_*[S] = \Phi \cap [M]$.

Proof. First, if we denote the inclusion $\kappa : N_S \to M$, $i : S \to M$ and the retraction $r : N_S \to S$. Then $i_* \circ r_* = k_*$ on homology. We have

$$\Phi \cap [M] = u \cap [M, M\backslash S] = u \cap k_*[N, N\backslash S] = i_* \circ r_*(k^* u \cap [N, N\backslash S]).$$

Thus $\Phi \cap [M]$ is i_* image of some element of $H_{n-k}(S)$. In other words, $\Phi \cap [M] = t \cdot i_*[S]$. We must show $t = 1$. We only need to prove it in N. For any point $p \in S$, we choose a neighborhood $U \subset S$, such that $(V, U) = (\mathbb{R}^n, \mathbb{R}^{n-k})$, where $V = \pi^{-1}(U)$ and $\pi : N \to S$ is the normal bundle. Then by definition of fundamental class, $[M]$ and $[S]$ restrict to fundamental classes μ_0 and ν_0 of $\mathbb{R}^n$ and $\mathbb{R}^{n-k}$ respectively. And u is pulled back to the Thom class u_0 of $\mathbb{R}^{n-k}$ in $\mathbb{R}^n$. Let $g : V \to N$ be the inclusion. Then if we read the relation in $H_n(N, N\backslash Q)$ where $Q \subset V$ is the set corresponding to $\mathbb{R}^k$, we have

$$i_*[S] = g_* \nu_0 = g_*(g^* u \cap \mu_0) = u \cap g_*(\mu_0) = u \cap [M] = t \cdot i_*[S].$$

The second inequality is because of the elementary relation $u_0 \cap \mu_0 = \nu_0$ in trivial bundle case in Step 1 of the Thom isomorphism. Hence $t = 1$. □

Now we could finish the proof of Equation (4.1) by virtue of Proposition 4.6.2. $\Phi_x \cap [M]$ is just $\epsilon(x)$ for any $x \in X \cap Y$ where the coorientation is induced from that of X and Y. (It is an exercise to check it. Notice we use normal bundle for the first and use tangent bundle for the second.) Hence the proof is reduced to prove

Proposition 4.6.3. $\Phi_{X\cap Y} = \Phi_X \cup \Phi_Y$. *Equivalently,* $i_*[X \cap Y] = i_*[X] \cdot i_*[Y]$.

Proof. It is helpful to introduce some self-explained super and sub scripts to our Thom classes and inclusions.

$$\begin{aligned}\Phi_{X\cap Y} \cap [M] =& (i^M_{X\cap Y})_*[X \cap Y]\\ =& (i^M_Y)_*(i^Y_{X\cap Y})_*[X \cap Y]\\ =& (i^M_Y)_*(u^Y_{X\cap Y} \cap [Y])\\ =& (i^M_Y)_*((i^M_Y)^* u^M_X \cap [Y])\\ =& \Phi_X \cap (i^M_Y)_*[Y]\\ =& \Phi_X \cap (\Phi_Y \cap [M])\\ =& (\Phi_X \cup \Phi_Y) \cap [M]\end{aligned}$$

□

Thom isomorphism has lots of interesting applications, e.g. Lefschetz fixed point theorem. Let us mention another: Euler class.

Definition 4.6.4. *Let* $E \to M$ *be a vector bundle and* $o : M \to E$ *be the zero section. Then the Euler class* $e(E)$ *is the image of Thom class* u_E *under the composition*

$$H^n(E, E_0) \to H^n(E) \xrightarrow{o^*} H^n(M).$$

For the special case when S is a submanifold of M and E is taken as its normal bundle. The Euler class $e(S)$ is the pull back of the Thom class through

$$i : S \to (M, M\backslash S).$$

In this case, Euler class could be viewed as the obstruction of deforming S to $M\backslash S$.

Proposition 4.6.5. *If the inclusion* $S \subset M$ *is homotopic to a map* $f : S \to M$ *whose image is contained in* $M\backslash S$*, then* $e(S) = 0$.

Proof. Hence the inclusion $i : S \to (M, M\backslash S)$ is homotopic to a map $\phi : S \to (M, M\backslash S)$ which is factored through $(M\backslash S, M\backslash S)$. Hence in cohomology $i^* = \phi^*$ factors through $H^*(M\backslash S, M\backslash S) = 0$. So $i^* = 0$ and hence $e(N) = 0$. □

Proposition 4.6.6. *$T(e(S)) = u \cup u$ where $u \in H^k(M, M\backslash S)$ is the Thom class.*

Proof. $T(e) = T(i^*(u)) = \pi^* i^*(u) \cup u = u \cup u$. □

Corollary 4.6.7. *If the codimension k of S in M is odd, then $2e(S) = 0$.*

The name of Euler class is from the following

Theorem 4.6.8. *Let M be an oriented n-manifold, $\Delta : M \to M \times M$ be the diagonal map. Then*

1. *$< e(M), [\Delta_M] >= \chi(M)$ where χ is the Euler characteristic.*
2. *The normal bundle of the diagonal (Δ_M) is isomorphic to TM.*

The Euler class is a characteristic class in the following sense

Definition 4.6.9. *A cohomology class $c(E) \in H^*(M)$ associated to any vector bundle $E \to M$ is called a characteristic class if it is natural with respect to pullbacks in the sense that $c(f^*E) = f^*(c(E))$.*