Quadratic Chabauty and L-functions

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ENS de Lyon

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Plan of the talk

Motivation: rational points on modular curves

- Finding rational points on curves
- Images of Galois representations associated to elliptic curves
- Chabauty method in the context of modular curves

The new input of "quadratic Chabauty"

- What is the "quadratic Chabauty" method ?
- Applying the method to families of modular curves

Nonvanishing of derivatives of modular L-functions

- Notations for modular L-functions
- Weighted sums: exact expression and asymptotic values
- Improving the estimates to get a computable range

Hypotheses and notations

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- For $O \in C(\mathbb{Q})$ fixed, $\iota : C \to J$ is the Albanese morphism sending O to 0.
- We assume $g \geq 2$ so that $C(\mathbb{Q})$ is *finite* by Faltings theorem.

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Chabauty's idea

Consider, for a prime p, the following commutative diagram

$$\begin{array}{ccc} C(\mathbb{Q}) & \overset{\iota}{\hookrightarrow} & J(\mathbb{Q}) \\ \downarrow & & \downarrow \\ C(\mathbb{Q}_p) & \overset{\iota}{\hookrightarrow} & J(\mathbb{Q}_p) \end{array}$$

In the p-adic variety $J(\mathbb{Q}_p)$,

$$C(\mathbb{Q}) \subset C(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}.$$

If $\operatorname{codim} \overline{J(\mathbb{Q})} \geq 1$, this should enable to prove finiteness !

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Proposition (Chabauty)

For any nonempty open subset $U \subset C(\mathbb{Q}_p)$, $\operatorname{Vect}_{\mathbb{Q}_p} \log(\iota(U)) = T_0 J_{\mathbb{Q}_p}$.

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Theorem (Chabauty)

If $r < g$ (Chabauty condition), then $C(\mathbb{Q})$ is finite.

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Recall the canonical identifications and pairing

$$(T_0 J_{\mathbb{Q}_p})^* \cong H^0(J_{\mathbb{Q}_p}, \Omega^1) \cong H^0(C_{\mathbb{Q}_p}, \Omega^1), \quad \langle \cdot, \cdot \rangle : T_0 J_{\mathbb{Q}_p} \times (T_0 J_{\mathbb{Q}_p})^*. \to \mathbb{Q}_p$$

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Definition (p-adic integration)

There is an analytic integration pairing

$$\begin{array}{ccc} J(\mathbb{Q}_p) \times H^0(C_{\mathbb{Q}_p}, \Omega^1) & \longrightarrow & \mathbb{Q}_p \\ (D, \omega) & \longmapsto & \int_D \omega := \langle \log D, \omega \rangle \end{array} .$$

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If C has a good reduction $C_{\mathbb{F}_p}$ at p and z is a well-chosen parameter at O, for $\omega = (\sum_{n \geq 0} a_n z^n) dz$ and any P reducing to O modulo p,

$$\int_O^P \omega := \int_{\iota(P)} \omega = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} z(P)^{n+1}.$$

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Theorem (Coleman)

Under the Chabauty condition $r < g$, if $p > 2g$,

$$\# C(\mathbb{Q}) \leq \# C_{\mathbb{F}_p}(\mathbb{F}_p) + (2g - 2).$$

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The Mordell-Weil sieve

Assume for simplicity $J(\mathbb{Q}) = \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_r$. For every good prime p, the commutative diagram

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gives, through $W_p = \iota(C(\mathbb{F}_p))$, congruence conditions on the coordinates $(n_1, \cdots, n_r)$ of elements of $\iota(C(\mathbb{Q}))$ modulo N_p the exponent of $J(\mathbb{F}_p)$.

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Hope for success of Mordell-Weil sieve + Chabauty

Find a finite set of primes S such that $C(\mathbb{Q}) \to \prod_{p \in S} C(\mathbb{F}_p)$ is injective (by Chabauty) and the only coordinates $(n_1, \cdots, n_r)$ satisfying congruences conditions modulo all N_p come from points of $C(\mathbb{Q})$ already known.

Galois representations associated to an elliptic curve

For an elliptic curve E over $\mathbb{Q}$ and a prime number p, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the p-torsion $E[p]$ defines a *Galois representation*

$$\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}).$$

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Is there a constant $C > 0$ such that for every prime $p > C$ and every E over $\mathbb{Q}$ without CM, $\rho_{E,p}$ is *surjective* ?

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- *Normaliser of split Cartan* (pair of distinct cyclic subgroups of order p).
- *Normaliser of nonsplit Cartan* (semi-linear action with respect to a $\mathbb{F}_{p^2}$-linear structure on $E[p]$).

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Modular curves are curves who (except their cusps) parametrise isomorphism classes of elliptic curves E together with a finite structure on E.

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Three families of modular curves: $X_0(p)$ for Borel, $X_{\mathrm{sp}}^+(p)$ (resp. $X_{\mathrm{nsp}}^+(p)$) for normaliser of split (resp. nonsplit Cartan).
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If $\mathrm{Im}\,\rho_{E,p}$ is in the Borel case, E defines a noncuspidal rational point on $X_0(p)$, and similary for the other cases.

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Restatement of Serre's uniformity conjecture

For any prime $p > C$, the modular curves $X_0(p)$, $X_{\mathrm{sp}}^+(p)$ and $X_{\mathrm{nsp}}^+(p)$ have no noncuspidal non-CM rational points.

Chabauty method in the context of modular curves

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Fundamental remark

Chabauty's theorem (and Coleman's method) still hold under the weaker hypothesis

$$\operatorname{rank} A(\mathbb{Q}) < \dim A$$

for some quotient abelian variety A of J, in particular if $A(\mathbb{Q})$ is finite (i.e. A is a *rank zero quotient*).

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Mazur's method (roughly)

If $J_0(p)$ has a rank zero quotient, if $\operatorname{Im}\rho_{E,p} \subset$ Borel, the associated point of $X_0(p)$ *never* reduces to a cusp hence $j(E) \in \mathbb{Z}$. The same thing holds for $J^+_{\mathrm{sp}}(p)$ and $J^+_{\mathrm{nsp}}(p)$.

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The two families to study

We will focus now on $X_{\mathrm{nsp}}^{+}(p)$ and $X_0(p)^{+} = X_0(p)/\langle w_p \rangle$ (whose jacobian is isogenous to $J_0(p)^{+}$).

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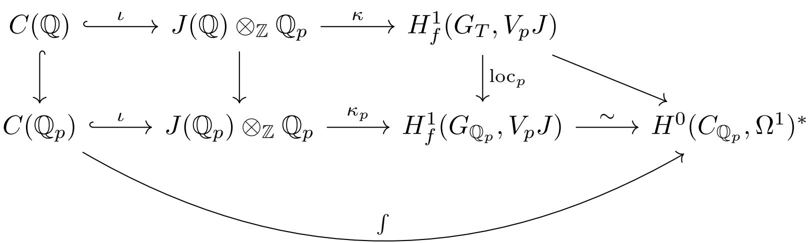

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$$\begin{array}{ccccccc} C(\mathbb{Q}) & \overset{\iota}{\hookrightarrow} & J(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p & \xrightarrow{\kappa} & H^1_f(G_T, V_pJ) & & \\ \downarrow & & \downarrow & & \downarrow \mathrm{loc}_p & \searrow & \\ C(\mathbb{Q}_p) & \overset{\iota}{\hookrightarrow} & J(\mathbb{Q}_p) \otimes_{\mathbb{Z}} \mathbb{Q}_p & \xrightarrow{\kappa_p} & H^1_f(G_{\mathbb{Q}_p}, V_pJ) & \xrightarrow{\sim} & H^0(C_{\mathbb{Q}_p}, \Omega^1)^* \end{array}$$

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Kim's idea

Replace V_pJ by a unipotent p-adic Lie group $U \twoheadrightarrow V_pJ$ over $\mathbb{Q}_p$,

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$$\begin{array}{ccc} C(\mathbb{Q}) & \xrightarrow{\kappa_U} & \mathrm{Sel}(U) \\ \big\downarrow & & \big\downarrow \mathrm{loc}_p \\ C(\mathbb{Q}_p) & \xrightarrow{\kappa_{U,p}} & H^1_f(G_{\mathbb{Q}_p}, U) \end{array} \qquad (\subset H^1_f(G_T, U))$$

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  where κ_U and $\kappa_{U,p}$ are Kummer maps, $\mathrm{Sel}(U)$ and $H^1_f(G_{\mathbb{Q}_p}, U)$ have variety structures and loc_p is algebraic.
- The map $\kappa_{U,p}$ has locally Zariski-dense image everywhere.
- The map loc_p is not dominant.

Then, $C(\mathbb{Q}) \hookrightarrow \kappa_{U,p}^{-1}(\mathrm{Im}\,\mathrm{loc}_p)$ which proves it is finite !

Quadratic Chabauty: the main theorem

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Definition (Néron-Severi group)

Let $\mathrm{NS}(J) := \mathrm{Pic}\, J / \mathrm{Pic}^0\, J$ be the Néron-Severi group of J. It is a finite type $\mathbb{Z}$-module, of rank denoted by $\rho = \rho(J)$.

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Theorem(Balakrishnan, Dogra)

One can find a group U satisfying the first two conditions, and

$$\dim \mathrm{Sel}(U) \leq r = \mathrm{rank}\, J(\mathbb{Q}), \quad \dim H^1_f(G_{\mathbb{Q}_p}, U) \geq g + \rho - 1.$$

Therefore, under the *quadratic Chabauty condition*

$$r < g + \rho - 1,$$

one has proved the finiteness of $C(\mathbb{Q})$!

Applications of the method

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Theorem (Balakrishnan, Dogra, Müller, Tuitman, Vonk)

The set of rational points of $X_{\mathrm{nsp}}^{+}(13)$ (for which $r = g = \rho = 3$) is made up with CM points and $\#X_{\mathrm{nsp}}^{+}(13)(\mathbb{Q}) = 7$.

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- Equation(s) for the curve.
- Iterated p-adic integrals for 1-forms on the curve to give explicit equations for the rational points.
- Mordell-Weil sieve to exclude all other possibilities.
- Special working case : $r = g$, $\rho > 1$.

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Reasonable working scopes

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WIP (Dogra, Vonk)

The quadratic Chabauty method also applies for C if

$$\operatorname{rank} A(\mathbb{Q}) < \dim A + \rho(A) - 1$$

for A a quotient abelian variety of J, in particular if $\operatorname{rank} A(\mathbb{Q}) = \dim A$ and $\rho(A) > 1$.

What is special about modular curves

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Theory of Eichler-Shimura

- If $f = \sum_{n=1}^{+\infty} a_n q^n$ is a newform of $S_2(\Gamma_0(N))$, $K_f := \mathbb{Q}(\{a_n\})$ is a totally real number field and there is a quotient A_f of $J_0(N)^{\mathrm{new}}$ of dimension $[K_f : \mathbb{Q}]$ with $\mathrm{End}(A_f) \otimes \mathbb{Q} = K_f$.

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- We have the decomposition
$$J_0(N)^{+,\mathrm{new}} \sim \bigoplus_f A_f$$
where f runs through representatives of the orbits of newforms of $S_2(\Gamma_0(N))^+$ by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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Fundamental remark for modular curves

As $\mathrm{NS}(A_f) \otimes \mathbb{Q} \cong K_f$ here (Pyle), for $J_0(N)^+$, it is enough to find either:

(a) One newform f such that $\mathrm{rank}\, A_f(\mathbb{Q}) = \dim A_f \geq 2$.

(b) Two newforms f such that $\mathrm{rank}\, A_f(\mathbb{Q}) = \dim A_f$.

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For $N = p$ or p^2 large enough, prove option (a) or (b).

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The rank part of BSD conjecture

For any abelian variety A over $\mathbb{Q}$, $\operatorname{rank} A(\mathbb{Q}) = \operatorname{ord}_{s=1} L(A, s)$.

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For any modular form f in $S_2(\Gamma_0(N))$, the L-function of f is defined for $\operatorname{Re}(s) > 2$ by

$$L(f, s) = \sum_{n=1}^{+\infty} \frac{a_n(f)}{n^s}.$$

It extends holomorphically to $\mathbb{C}$ and $L(f, 1) = 0$ if $f \in S_2(\Gamma_0(N))^+$.

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It extends holomorphically to $\mathbb{C}$ and $L(f, 1) = 0$ if $f \in S_2(\Gamma_0(N))^+$. If f is a newform,

$$L(A_f, s) = \prod_{g \sim f} L(g, s)$$

where g goes through the $[K_f : \mathbb{Q}]$ newforms $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate to f.

What to prove analytically

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Theorem (Kolyvagin-Logachev)

For f a newform in $S_2(\Gamma_0(N))$, if $\mathrm{ord}_{s=1} L(f,s) = k \in \{0,1\}$ then A_f satisfies the rank part of BSD conjecture, i.e.

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Restated objective

For any $N = p$ or p^2 large enough, prove:
There are at least two newforms $f \in S_2(\Gamma_0(N))^+$ such that $L'(f,1) \neq 0$.

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$$L'(f,1) = 2\sum_{n=1}^{+\infty} \frac{a_n(f)}{n} E_1\left(\frac{2\pi n}{\sqrt{N}}\right)$$

where $E_1(y) = \int_y^{+\infty} e^{-t}/t dt$ is the exponential integral function.

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Main idea for computations

To prove that there is one f such that $L'(f, 1) \neq 0$, it is enough to prove that a weighted sum of the $L'(f, 1)$ is nonzero !

Notations for the weighted sums

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- For any linear forms A, B on $S_2(\Gamma_0(N))$,

$$\langle A, B \rangle_N = \sum_f \frac{\overline{A(f)}B(f)}{\|f\|^2}$$

where f runs through a Petersson-orthogonal basis of $S_2(\Gamma_0(N))$ with superscripts $+, -, \mathrm{new}$ added for the corresponding subspaces of $S_2(\Gamma_0(N))$.

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Lemma

For any m prime to p,

$$\langle a_m, L'\rangle_{p^2}^{+,\mathrm{new}} = \langle a_m, L'\rangle_{p^2}^{+} - \frac{1}{p-1}\left(\langle a_m, L'\rangle_p^{+} + \frac{\ln(p)}{2}\langle a_m, L\rangle_p^{-}\right)$$

so it is enough to compute only $\langle a_m, L'\rangle_N^{+}$ and $\langle a_m, L\rangle_p^{-}$.

Our main tool: Petersson trace formula

Proposition (Restricted Petersson trace formula)

For any integers $m, n, N \geq 1$ *:*

$$\frac{\langle a_m, a_n\rangle_N^+}{2\pi\sqrt{mn}} = \delta_{mn} - 2\pi\left(\sum_{N|c}\frac{S(m,n;c)}{c}J_1\left(\frac{4\pi\sqrt{mn}}{c}\right)\right) - 2\pi\left(\sum_{(d,N)=1}\frac{S(m,nN^{-1};d)}{d\sqrt{N}}J_1\left(\frac{4\pi\sqrt{mn}}{d\sqrt{N}}\right)\right)$$

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$$S(m,n;c) = \sum_{k \in (\mathbb{Z}/c\mathbb{Z})^*} e^{2i\pi(mk+nk^{-1})/c}$$

is the Kloosterman sum.

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$$\frac{\langle a_m, L' \rangle_N^+}{4\pi} = E_1\left(\frac{2\pi m}{\sqrt{N}}\right) - 2\pi\sqrt{m}\left(\sum_{N|c} \frac{\mathcal{S}(c)}{c} + \sum_{(d,p)=1} \frac{\mathcal{T}(d)}{d\sqrt{N}}\right),$$

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and a similar formula holds for $\langle a_m, L\rangle_N^-$.

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Remark

For $m \ll \sqrt{N}$, the main term is $E_1(2\pi m/\sqrt{N}) \sim \ln(N)/2$ hence $\langle a_m, L'\rangle_N^+ \sim 2\pi\ln(N)$.

First estimates: Weil bounds

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First, one has $|J_1(x)| \le |x|/2$, and

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Consequence

For $m \ll \sqrt{N}$,

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the (effective) error terms coming respectively from the $\mathcal{S}(c)$ and $\mathcal{T}(d)$.

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For $N = p$, it is enough to prove that $\langle a_1, L'\rangle_p^+ \neq 0$ and $\langle a_2, L'\rangle_p^+ / \langle a_1, L'\rangle_p^+ \notin \mathbb{Z}$, and similarly for $N = p^2$.

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Proof.

When $\langle a_1, L'\rangle_p^+ \neq 0$, the only situation when option (a) is not satisfied is when only one newform f in the basis satisfies $L'(f, 1) \neq 0$, and then

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For $N = p$, it is enough to prove that $\langle a_1, L' \rangle_p^+ \neq 0$ and $\langle a_2, L' \rangle_p^+ / \langle a_1, L' \rangle_p^+ \notin \mathbb{Z}$, and similarly for $N = p^2$.

Proof.

When $\langle a_1, L' \rangle_p^+ \neq 0$, the only situation when option (a) is not satisfied is when only one newform f in the basis satisfies $L'(f,1) \neq 0$, and then

$$\frac{\langle a_2, L' \rangle_p^+}{\langle a_1, L' \rangle_p^+} = \frac{a_2(f) L'(f,1)}{\|f\|^2} \frac{\|f\|^2}{L'(f,1)} = a_2(f).$$

Now, if $a_2(f) \notin \mathbb{Z}$, $K_f \neq \mathbb{Q}$ so f has nontrivial conjugates g such that $L'(g,1) \neq 0$ as well, contradiction. □

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Proposition

After improving the bounds specifically for $m = 1$ *and* $m = 2$*, one finds*

$$\begin{array}{rll|rll}
\langle a_1, L'\rangle_p^+ > 0 & \textit{for} & p \geq 1213 & \langle a_1, L'\rangle_{p^2}^{+,new} > 0 & \textit{for} & p \geq 47 \\
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Remarks to improve this result

- Those bounds are still too large to be complemented by computer.
- The term $O(m/\sqrt{N})$ coming from the $\mathcal{T}(d)$ needs to be improved.
- The Kloosterman sums oscillate a lot.

Pólya-Vinogradov-like inequality for Kloosterman sums

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Proposition

For every $d > 1$, every k invertible modulo d and every $m, K, K' \in \mathbb{N}$,

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As $J_1(x) \approx x/2$ for x small, for $d > 1$,

$$\begin{aligned} |\mathcal{T}(d)| &\lessapprox \frac{2\pi\sqrt{m}}{d\sqrt{p}} \sum_{n=1}^{+\infty} S(1, nN^{-1}; d) E_1 \left(\frac{2\pi n}{\sqrt{N}} \right) \\ &\lessapprox \frac{8}{\pi} \frac{\sqrt{m}}{\sqrt{N}} (\log(d) + 1.5) E_1 \left(\frac{2\pi}{\sqrt{N}} \right) \end{aligned}$$

by Abel transform, to be compared to the bound $\tau(d)/\sqrt{d}$ coming from the Weil bounds.

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- Infinite families of jacobians satisfying quadratic Chabauty.
- Devise a "quadratic Mazur's method".