You should attempt all the questions on this sheet, but questions 1, 2, 3 will marked for credit, and must be handed in by 3pm Friday, week 9.

(1) Which of the following are lattices in \( \mathbb{Z}^2 \)? What is the index?
   (i) \( \{(x, y) \in \mathbb{Z}^2 : x + y = 1\} \).
   (ii) \( \{(x, y) \in \mathbb{Z}^2 : x + y = 0\} \).
   (iii) \( \{(x, y) \in \mathbb{Z}^2 : 2 \mid x\} \).
   (iv) \( \{(x, y) \in \mathbb{Z}^2 : x \equiv y \pmod{3}\} \).
   (v) \( \{(x, y) \in \mathbb{Z}^2 : x \equiv y \pmod{3}, x \equiv 2y \pmod{5}\} \).

(2) Which of the following are convex? Which of the following are symmetric?
   (i) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 0\} \).
   (ii) \( \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 < 1\} \).
   (iii) \( \{(x, y, z) \in \mathbb{R}^3 : 3x^2 + 5y^2 + 7z^2 < 1\} \).

(3) Let \( p \) be an odd prime satisfying \( \left( -\frac{2}{p} \right) = 1 \). Show that there are integers \( x, y \) such that \( x^2 + 2y^2 = p \).

(4) Find an odd prime \( p \) for which \( \left( -\frac{5}{p} \right) = 1 \) but which is not of the shape \( x^2 + 5y^2 \) with \( x, y \in \mathbb{Z} \).

(5) Let \( p \equiv 1 \pmod{3} \) be prime. Show that there is some \( f \in \mathbb{Z} \) such that \( f^2 + f + 1 \equiv 0 \pmod{p} \). Show that \( p = x^2 + xy + y^2 \) for some \( x, y \in \mathbb{Z} \).