You should attempt all the questions on this sheet, but questions Q2(i), Q3, Q5(i) will marked for credit, and must be handed in by 3pm Friday, week 7.

(1) List (and memorise!) the squares modulo 4, 8, 3, 5, 7.

(2) (i) Show that the sequence $n^5 - n + 3$ does not contain any squares. \(\text{(Hint: consider modulo 5.)}\)
(ii) Let $p$ be a prime $p \equiv 3, 5 \pmod{8}$. Show that the sequence $n! + n^p - n + 2$ contains at most finitely many squares.

(3) (i) Is 219 a square modulo 383?
(ii) Is 219 a square modulo 143? (Be careful!)

(4) For which primes is 5 a quadratic residue? For which primes is 3 a quadratic residue?

(5) Suppose $p, q$ are primes with $p = 2q + 1$.
   (i) Show that if $q \equiv 1 \pmod{4}$ then 2 is a primitive root modulo $p$.
   (ii) Under what conditions on $q$ is 5 a primitive root modulo $p$?

(6) Show that the equation $y^2 = x^3 + 7$ has no integral solutions. \(\text{(Hint: rewrite as } y^2 + 1 = x^3 + 8.\)

(7) Let $m$ be a positive odd integer. In this exercise we prove the identity
\[
\frac{\sin mx}{\sin x} = (-4)^{(m-1)/2} \prod_{t=1}^{(m-1)/2} \left( \sin^2 x - \sin^2 \frac{2\pi t}{m} \right),
\]
(a) By induction on $m$ (odd) show (simultaneously) that
\[
\frac{\sin mx}{\sin x} = f_m(\sin^2 x), \quad \frac{\cos mx}{\cos x} = g_m(\sin^2 x),
\]
where $f_m$ and $g_m$ are polynomials of degree $(m-1)/2$ with leading coefficient $(-4)^{(m-1)/2}$.
(b) Show that $\sin^2 \frac{2\pi t}{m}$ with $t = 1, 2, \ldots, (m-1)/2$ are distinct roots of $f_m$.
(c) Deduce the identity.