Explicit Arithmetic for Modular Curves

Exercises IV

(A) (Computational Exercise.) A point $P$ on a curve $C$ of genus $g$ is called a **Weierstrass point** if there is a regular differential $\omega \in \Omega(C)$ such that $\operatorname{ord}_P(\omega) \geq g$. Determine all $N \leq 100$ such that the $\infty$ cusp of $X_0(N)$ is a Weierstrass point.

(B) To do this exercise you need to a little about how to calculate valuations at points. If this is unfamiliar, perhaps skip this exercise.

(i) Let $X : y^2 = a_{2g+2}x^{2g+2} + \cdots + a_0$ be a curve of genus $g$ where $a_{2g+2} \neq 0$. Let $\infty_+$ be one of the two points at infinity. Show that

\[
\operatorname{ord}_{\infty_+}\left(\frac{dx}{y}\right) = g - 1, \quad \operatorname{ord}_{\infty_+}\left(\frac{x\,dx}{y}\right) = g - 2, \ldots, \operatorname{ord}_{\infty_+}\left(\frac{x^{g-1}\,dx}{y}\right) = 0.
\]

(ii) Let $X : y^2 = a_{2g+1}x^{2g+1} + \cdots + a_0$ be a curve of genus $g$ (here necessarily $a_{2g+1} \neq 0$ otherwise the genus would be smaller than $g$). Let $\infty$ be the unique point at infinity. Show that

\[
\operatorname{ord}_\infty\left(\frac{dx}{y}\right) = 2(g - 1), \quad \operatorname{ord}_\infty\left(\frac{x\,dx}{y}\right) = 2(g - 2), \ldots, \operatorname{ord}_\infty\left(\frac{x^{g-1}\,dx}{y}\right) = 0.
\]

(C) A basis for $S_2(\Gamma_0(64))$ is

\[
q - 3q^9 + O(q^{12}),
q^2 - 2q^{10} + O(q^{12}),
q^5 + O(q^{12})
\]

Deduce (very very quickly) that $X_0(64)$ is not hyperelliptic. (Hint: Use exercises (A), (B)).

Fun Fact: $X_0(64)$ is actually the Fermat quartic $x^4 + y^4 = z^4$. 