MA3D5 Galois Theory

Homework Assignment 4

The deadline is 2pm Thursday, week 9. Please hand in your solutions to Questions 1 and 2 to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Let \( L = \mathbb{Q}(\zeta, \sqrt[3]{2}) \) where \( \zeta = \exp(2\pi i/3) \). In the lectures we showed that \( L/\mathbb{Q} \) is Galois and identified its Galois group with \( S_3 \), by noting that \( L \) is the splitting field of \( f = x^3 - 2 \), and ordering the roots of \( f \) as \( \sqrt[3]{2}, \zeta \sqrt[3]{2}, \zeta^2 \sqrt[3]{2} \).
   (a) Give the following as subgroups of \( S_3 \):
   \( \mathbb{Q}(\sqrt[3]{2})^*, \mathbb{Q}(\zeta)^* \).
   (b) Calculate the following intermediate fields
   \( \{1, (1,2,3), (1,3,2)\}^\dagger, \{1, (2,3)\}^\dagger \).
   (c) With the help of the Fundamental Theorem of Galois Theory show that there are precisely six intermediate fields \( F \) for the extension \( L/\mathbb{Q} \) (including \( L, \mathbb{Q} \)), and identify the ones for which \( F/\mathbb{Q} \) is Galois.

2. Let \( L = \mathbb{Q}(\sqrt{-1}, \sqrt[3]{2}, \sqrt[3]{3}) \).
   (a) Show that \( [L: \mathbb{Q}] = 8 \). (Hint: you may use the fact that \( [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4 \) for distinct primes \( p, q \).)
   (b) Show that \( L/\mathbb{Q} \) is Galois, and compute its Galois group as a subgroup of \( S_6 \), by noting that \( L \) is the splitting field of \( f = (x^2 + 1)(x^2 - 2)(x^2 - 3) \) and ordering the roots of \( f \) as \( i, -i, \sqrt[3]{2}, -\sqrt[3]{2}, \sqrt[3]{3}, -\sqrt[3]{3} \).
   (c) Give the following as subgroups of \( S_6 \):
   \( \mathbb{Q}^*, \mathbb{Q}(\sqrt{-1})^*, \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{3})^*, L^* \).
   (d) Calculate the following intermediate fields:
   \( \{1, (3,4)\}^\dagger, \{1, (1,2)(5,6)\}^\dagger, \{1, (1,2), (5,6), (1,2)(5,6)\}^\dagger \).
   (e) Explain why \( F/\mathbb{Q} \) is Galois for all intermediate fields \( F \) of \( L/\mathbb{Q} \).

3. Let \( f \) be a squarefree separable polynomial over \( K \). Let \( L = K(\alpha_1, \ldots, \alpha_n) \) be the splitting field of \( f \) where \( \alpha_1, \ldots, \alpha_n \) are the roots of \( f \). Define the discriminant of \( f \) to be
   \[ D(f) = \left( \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j) \right)^2. \]
   (i) Show that \( D(f) \in K \).
   (ii) Show that \( D(f) \) is a square in \( K \) if and only if \( \text{Aut}(L/K) \subseteq A_n \).
   **Hint:** Revise alternating polynomials in your Introduction to Abstract Algebra notes.