MA3D5 Galois Theory

Homework Assignment 3

The deadline is 2pm Thursday, week 7. Please hand in your answers to Questions 3, 4, 5, 6 to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. (a) Let $M/K$ be a field extension. Let $L = \{ \alpha \in M : \alpha \text{ is algebraic over } K \}$.
   Show that $L$ is a field.
   (b) Let $\mathbb{Q}$ be the set of elements in $\mathbb{C}$ algebraic over $\mathbb{Q}$. Part (a) tells us that $\mathbb{Q}$ is a field. Now let $\beta \in \mathbb{C}$ be algebraic over $\mathbb{Q}$. Show that $\beta \in \mathbb{Q}$.

2. Let $L = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. In Example 69 in the lecture notes we computed $\text{Aut}(L/\mathbb{Q})$. Write down all its subgroups $H$, and compute the corresponding fixed fields $L^H$ (Hint: see Example 72).

3. Which of the following extensions are normal? Which are separable?
   (a) $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}$.
   (b) $\mathbb{Q}(\sqrt{7})/\mathbb{Q}$.
   (c) $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}(\sqrt{-7})$.
   (d) $K(t^{1/3})/K$ where $K = \mathbb{F}_3(t)$.

4. Let $L = \mathbb{Q}(\sqrt{2})$. Compute $\text{Aut}(L/\mathbb{Q})$ and $L^{\text{Aut}(L/\mathbb{Q})}$.

5. If $L/F$ and $F/K$ are Galois, does $L/K$ have to be a Galois extension? Prove or give a counterexample. (Big Hint: Consider $L$ as in question 4.)

6. Let $L$ be a subfield of $\mathbb{C}$ that is a finite Galois extension of $\mathbb{Q}$.
   (a) Let $\alpha \in L$ and let $\overline{\alpha}$ be its complex conjugate. Show that $\overline{\alpha} \in L$.
   (b) Let $\sigma : L \to L$, $\sigma(\alpha) = \overline{\alpha}$.
   Show that $\sigma \in \text{Aut}(L/\mathbb{Q})$, and has order 1 or 2.
   (c) Show moreover that $\sigma$ has order 2 if and only if $L \not\subset \mathbb{R}$.
   (d) Let $F = L^{\sigma}$. Show that $[L : F] = 1$ or 2 according to whether $L \subset \mathbb{R}$, $L \not\subset \mathbb{R}$.

7. Let $p$ be an odd prime. Let $\zeta = \exp(2\pi i/p)$.
   (a) Show that the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois.
   (b) Define $\mu : (\mathbb{Z}/p\mathbb{Z})^* \to \text{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q})$, $\mu(\overline{\alpha})(\zeta) = \zeta^a$.
   Show that $\mu$ is well-defined and is in fact an isomorphism.
   (c) Let $\sigma$ be as in part (c) of Question 6. Show that $\mathbb{Q}(\zeta)^{\sigma} = \mathbb{Q}(\zeta + 1/\zeta)$.