MA3D5 Galois Theory

Homework Assignment 2

The deadline is 2pm Thursday, week 5. Please hand in your answers to questions 2, 3 and 4 the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Let \( f \in \mathbb{Q}[x] \) be a polynomial of degree \( n \). Show that the splitting field of \( f \) has degree \( \leq n! \).

2. Let \( p, q \) be distinct primes.
   (a) Show that \( \sqrt{p} \notin \mathbb{Q}(\sqrt{q}) \).
   (b) Determine with proof the degree \([\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}]\).
   (c) Determine with proof the degree \([\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{pq}) : \mathbb{Q}]\).
   (d) Let \( g(x) = x^4 - 2(p+q)x^2 + (p-q)^2 \).
   Show that \( \sqrt{p} + \sqrt{q} \) is a root of \( g \). Deduce that \( g \) is irreducible. (Hint: use the fact \( \mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q}) \) which you proved in Assignment 1.)

3. Let \( f = x^3 + x + 3 \). In Assignment 1 you showed that \( f \) is irreducible, and that it has exactly one real root.
   (a) Let \( \theta \) be the real root of \( f \). Let \( \phi, \phi' \) be the two other roots. Compute
   \[
   [\mathbb{Q}(\theta) : \mathbb{Q}] \quad [\mathbb{Q}(\theta, \phi) : \mathbb{Q}] \quad [\mathbb{Q}(\theta, \phi, \phi') : \mathbb{Q}].
   \]
   (b) Without writing down the minimal polynomial for \( \theta^2 \), show that \( \mathbb{Q}(\theta^2) = \mathbb{Q}(\theta) \).
   (c) Write down the minimal polynomial for \( \theta^2 \).

4. Let \( L/K \) be a field extension with degree \([L : K] = p\) where \( p \) is a prime. Show that \( L/K \) is a simple extension.