MA3D5 Galois Theory

Homework Assignment 1

The deadline is **2pm Thursday, week 3.** Please hand in your answers to questions 5, 6 to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Show that \( f \) is irreducible over the given field \( K \):
   (a) \( f = x^5 + 4x^2 - 6 \) over \( \mathbb{Q} \).
   (b) \( f = x^5 + t^2x^2 - 3t \) over \( \mathbb{F}_5(t) \).
   (c) \( f = x^{p-1} + x^{p-2} + \cdots + 1 \) over \( \mathbb{Q} \), where \( p \) is a prime.

2. Let \( f = x^3 + x + 3 \).
   (a) Show that \( f \) is irreducible over \( \mathbb{Q} \).
   (b) Show that \( f \) has exactly one real root.

3. Let \( p \) be a prime. Show in \( \mathbb{F}_p[x, y] \) that
   \[(x + y)^p = x^p + y^p.\]

4. Let \( a = (1 + i)\mathbb{Z}[i] \). Show that \( \mathbb{Z}[i]/a \) is a field. How many elements does it have? Write down addition and multiplication tables for the elements.

5. Let \( p, q \) be distinct primes. Show that \( \mathbb{Q} (\sqrt{p} + \sqrt{q}) = \mathbb{Q} (\sqrt{p}, \sqrt{q}) \).

6. Compute and simplify the splitting fields of \( f \in K[x] \) over the given \( K \).
   (a) \( f = (x^2 + x + 1)(x^2 - 5) \), \( K = \mathbb{Q} \).
   (b) \( f = (x^2 + x - 1)(x^2 - 5) \), \( K = \mathbb{Q} \).
   (c) \( f = x^3 - 7 \), \( K = \mathbb{Q} \).
   (d) \( f = x^3 - 7 \), \( K = \mathbb{Q} (\sqrt{-3}) \).

7. (a) Let \( f \) be an irreducible quadratic polynomial over \( \mathbb{Q} \). Show that its splitting field has the form \( \mathbb{Q} (\sqrt{D}) \) where \( D \) is a squarefree integer \( \neq 0, 1 \).
   (b) Let \( f = x^3 - 3x + 1 \). Show that its splitting field over \( \mathbb{Q} \) is contained in \( \mathbb{R} \).

8. Very hard! Don’t spend too much time on this. Show that \( x^n + x + 3 \) is irreducible for all \( n \geq 2 \).

9. **Aptitude test for prospective university administrators** Reformulate the above questions and your answers in the new Warwick tone of voice.