Algebraic Number Theory
Example Sheet 4

Hand in the answers to questions 3, 6, 7. Deadline 2pm Thursday, Week 10.

1. Let \( a, b \) be ideals of \( \mathcal{O}_K \) with \( a \subseteq b \).
   (i) Show that \( \text{Norm}(a) \geq \text{Norm}(b) \).
   (ii) Show that \( \text{Norm}(a) = \text{Norm}(b) \) if and only if \( a = b \).

2. Let \( K \) be a number field. Show that \( \mathcal{O}_K \) is a PID if and only if it is a UFD.

3. Let \( K = \mathbb{Q}(\sqrt{-2}) \). Show that \( \mathcal{O}_K \) is a principal ideal domain. Deduce that every prime \( p \equiv 1, 3 \pmod{8} \) can be written as \( p = x^2 + 2y^2 \) with \( x, y \in \mathbb{Z} \).

4. Compute the class groups of the following quadratic fields
   \( \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-30}) \).

5. (i) Let \( \alpha, \beta \) be non-zero elements of \( \mathcal{O}_K \). Suppose \( \langle \alpha \rangle = \langle \beta \rangle \). Show that \( \alpha = \beta \varepsilon \) for some \( \varepsilon \in \mathcal{U}(K) \).
   (ii) Let \( a, b \) be non-zero ideals with \( a + b = \langle 1 \rangle \) (we say \( a, b \) are coprime). Show that \( a, b \) are coprime in the following sense: if \( \mathfrak{p} \) is a prime ideal then \( \mathfrak{p} \) divides at most one of \( a, b \).
   (iii) Let \( a, b \) be coprime non-zero ideals. Suppose \( ab = c^n \) where \( c \) is an ideal and \( n \) is a positive integer. Show that there are ideals \( c_1, c_2 \) such that
      \[ a = c_1^n, \quad b = c_2^n, \quad c = c_1c_2. \]
   (iv) Give a counterexample, with \( K = \mathbb{Q} \), to show that (iii) fails if \( a, b \) are not coprime.
   (v) Let \( x, y \in \mathbb{Z} \) and satisfy \( x^2 + 2 = y^3 \). Show that \( x, y \) are odd, and deduce that the ideals \( a = \langle x + \sqrt{-2} \rangle, b = \langle x - \sqrt{-2} \rangle \) are coprime.
   (vi) Continuing from (v), show carefully that \( x + \sqrt{-2} = (u + v\sqrt{-2})^3 \) for some \( u, v \in \mathbb{Z} \). Hence determine the solutions to \( x^2 + 2 = y^3 \) with \( x, y \in \mathbb{Z} \).

6. Let \( K = \mathbb{Q}(\sqrt{-5}) \).
   (a) Show that \( \text{Cl}(K) \cong C_2 \).
   (b) Let \( a \) be an ideal of \( \mathcal{O}_K \) and suppose \( a^3 \) is principal. Show that \( a \) is principal.
   (c) Solve \( x^2 + 5 = y^3 \) with \( x, y \in \mathbb{Z} \).

7. Let \( K = \mathbb{Q}(\sqrt{2}) \). You may suppose that \( 1, \sqrt{2}, \sqrt{2^2} \) is an integral basis for \( \mathcal{O}_K \). Show that
   \[ \mathcal{U}(K) = \{ \pm (\sqrt{2} - 1)^n : n \in \mathbb{Z} \} \]
   You may need to use WolframAlpha, MATLAB or a similar package to compute approximations to the embeddings of some algebraic numbers.