Global rigidity of random graphs in $\mathbb{R}$

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Abstract

Consider the Erdős-Rényi random graph process $\{G_m\}_{m\geq 0}$ in which we start with an empty graph G_0 on the vertex set $[n]$, and in each step form G_i from G_{i-1} by adding one new edge chosen uniformly at random. Resolving a conjecture by Benjamini and Tzalik, we give a simple proof that w.h.p. as soon as G_m has minimum degree 2 it is globally rigid in the following sense: For any function $d\colon E(G_m) \to \mathbb{R}$, there exists at most one injective function $f\colon [n] \to \mathbb{R}$ (up to isometry) such that $d(ij) = |f(i) - f(j)|$ for every $ij \in E(G_m)$. We also resolve a related question of Girão, Illingworth, Michel, Powierski, and Scott in the sparse regime for the random graph and give some open problems.

1 Introduction

Let $V \subseteq \mathbb{R}^d$ be a finite set of distinct points, and suppose we only know distances between some of them. The pairs of points with known distances naturally form a graph G on the vertex set V. Which properties of G are sufficient for the unique reconstruction of V, up to isometry? When $d \geq 2$ one needs to impose further restrictions on V, as otherwise there are examples which show that if G is missing just one (carefully chosen) edge, a unique reconstruction is not possible. For example, consider the configuration with $n-2$ points on a line and two points outside of the line. Then we cannot decide whether these two points lie on the same side of the line or not, unless we are given the distance between them. It turns out that if one restricts the coordinates of V to be algebraically independent over rationals, then whether or not V is reconstructible from G depends on combinatorial properties of G. This case has been extensively studied (e.g. see [3, 9, 10, 11, 12, 14]; for a thorough introduction to the topic, see [13]).

Recently, Benjamini and Tzalik [4] studied what happens when $V \subseteq \mathbb{R}$. It is a folklore result that if the known-distance graph G is 2-connected (e.g. see [13, Chapter 63]) and V is algebraically independent, then one can uniquely reconstruct V. However, unlike in the case of higher dimensions, there are no clear obstacles which justify the necessity of algebraic independence in the 1-dimensional case. Indeed, the main result of Benjamini and Tzalik [4] states that for any given V if the graph of known distances is distributed as an Erdős-Rényi random graph $G \sim G(n,p)$ for $p \geq C \log n/n$, then with high probability (w.h.p.) V is reconstructible from G. This was strengthened by Girão, Illingworth, Michel, Powierski, and Scott [8] to a *hitting time* result, which we now state.

Consider a random graph process $\{G_m\}_{m\geq 0}$ on the vertex set V, where G_0 is an empty graph and each G_i is formed from G_{i-1} by adding a new edge uniformly at random. Let $\tau := \tau_2$ denote the smallest m such that $\delta(G_m) \geq 2$. We say that two functions $f, f'\colon [n] \to \mathbb{R}$ are isometric if there exists $b \in \mathbb{R}$ and $a \in \{1, -1\}$ such that $f = af' + b$.

Definition 1.1. Given a function $f\colon [n] \to \mathbb{R}$ and graph G on vertex set $[n]$, we define the *G-distance function of f*, denoted $d_{f,G} = d$, by $d(ij) := |f(i) - f(j)|$ for every $ij \in E(G)$. We also say that a function $f\colon [n] \to \mathbb{R}$ *realizes* a function $d\colon E(G) \to \mathbb{R}$ if $d = d_{f,G}$.

We formulate the theorem of Girão et al. in this terminology.

Theorem 1.2 ([8])**.** *Let $f\colon [n] \to \mathbb{R}$ be an injective function. Then in the random graph process $\{G_m\}_{m\geq 0}$ on the vertex set $[n]$, w.h.p. f is the unique (up to isometry) function that realizes d_{f,G_τ}.*

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Given a connected graph G and a function $d\colon E(G) \to \mathbb{R}^+$, we can find an f which realizes d as follows. For each permutation π of $[n]$, we set $f_\pi(1) = 0$ and $U = \{1\}$, and as long as $U \neq [n]$ find an edge $uv \in G$ with $u \in U$ and $v \notin U$, and set $f_\pi(v) := f_\pi(u) + \text{sign}(\pi(v) - \pi(u))d(uv)$. At the end we simply check whether f_π realizes $d = d_{f,G_\tau}$. Assuming there is a unique f (up to isometry) which realizes d, which is the case in Theorem 1.2, this procedure is guaranteed to find it.

It is important to observe that, in Theorem 1.2, we are first given V and then we construct a graph of known distances. Benjamini and Tzalik [4] conjectured that a stronger version should also hold, namely that G_τ not only reconstructs the given V, but in fact it can reconstruct every V. This property is known as *global rigidity*. Note that minimum degree 2 is necessary as the embedding of a vertex of degree 1 can be ambiguous. We resolve the conjecture in the affirmative by showing the following.

Theorem 1.3. *In the random graph process $\{G_m\}_{m\geq 0}$ on the vertex set $[n]$, w.h.p. G_τ has the following property: For every function $d\colon E(G_\tau) \to \mathbb{R}^+$, up to isometry there exists at most one injective function $f\colon [n] \to \mathbb{R}$ which realizes d. In particular, G_τ is globally rigid.*

Note that in Theorem 1.3 we do not impose any restriction on d, and it very well may be that no injective function f satisfies the desired property. In the case where d comes from a given embedding of $[n]$ in $\mathbb{R}$, we know that f is a unique function which realizes d_{f,G_τ}.

As discussed earlier, for sparser random graphs one cannot hope for a unique function which realizes every d. However, Girão et al. [8] showed that given an injective $f\colon [n] \to \mathbb{R}$, the Erdős-Rényi random graph $G(n,p)$ with $p = \omega(1/n)$ contains w.h.p. a subset $V' \subseteq [n]$ such that $f' = f|_{V'}$ is the unique function which realizes $d_{f',G[V']}$. They asked if $1/n$ is a threshold for the property that $G(n,p)$ uniquely reconstructs a constant fraction of vertices for *any* injective function f. We show that this is indeed the case. Moreover, we show that we can always reconstruct the same set of vertices, the size of which is a fraction of n arbitrarily close to 1. For simplicity, we work with the $G(n,m)$ random graph model (the equivalent statement for $G(n,p)$ follows by [7, Theorem 1.4]), where a graph is chosen uniformly at random among all labeled graphs with n vertices and m edges.

Theorem 1.4. *For every $\varepsilon > 0$ there exists $C > 0$ such that the following holds. Let $G \sim G(n,m)$ for $m \geq Cn$. Then w.h.p. there exists a subset $V' \subseteq V(G)$ of size $|V'| \geq (1-\varepsilon)n$ such that the induced subgraph $G' = G[V']$ has the following property: For every function $d\colon E(G') \to \mathbb{R}^+$, up to isometry there exists at most one injective function $f\colon V' \to \mathbb{R}$ which realizes d.*

We prove Theorems 1.3 and 1.4 in Section 2, before finishing with some open problems in Section 3.

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2 Proof

The following lemma is the crux of our proofs. Both Theorem 1.3 and Theorem 1.4 are then derived as easy corollaries from it.

Lemma 2.1. *Let G be a graph with $V(G) = [n]$, and suppose it satisfies the following two properties:*

(P1) For every disjoint $U, W \subseteq V(G)$ of size $|U|, |W| \geq n/15$ there is an edge in G between U and W.

(P2) For every $U \subseteq V(G)$ of size $n/15 \leq |U| < n$, there exists a vertex $v \in V(G) \setminus U$ with at least two neighbors in U.

Then, every distance function $d\colon E(G) \to \mathbb{R}^+$ is realizable (up to isometry) by at most one injective function.

Proof. Let f and $g : [n] \to \mathbb{R}$ be two injective functions which realize d. Let

$$L_f := \big\{i \in [n] : |\{x \in [n] : f(i) < f(x)\}| \geq \lceil n/2 \rceil\big\}$$
$$R_f := \big\{i \in [n] : |\{x \in [n] : f(x) < f(i)\}| \geq \lceil n/2 \rceil\big\}$$

be the *left-half* and *right-half* of f (omitting the middle vertex when n is odd), and define L_g and R_g analogously.

We can assume without loss of generality that $|L_f \cap L_g| \geq \lceil (|L_f| - 1)/2 \rceil \geq \lceil (n-3)/4 \rceil > n/5$ (otherwise we consider instead the function $-g$, which is isometric to g, and use that $L_{-g} = R_g$), where we note that the result is trivial if $n = 1$, so that we can assume $n > 1$ and hence, from (P2), that $n > 15$. Then $|R_f \cap R_g| = |R_f| + |R_g| - |R_f \cup R_g| \geq n - 1 - |\overline{(L_f \cap L_g)}| \geq n/5$. Set $L := L_f \cap L_g$ and $R := R_f \cap R_g$.

We first prove that the induced bipartite graph $G[L, R]$ contains a sufficiently large connected component. Let $C^{(1)}, \dots, C^{(k)}$ be any ordering of the connected components of $G[L, R]$. Toward a contradiction, assume that each connected component contains at most $n/15$ vertices, i.e. for all $j \in [k]$, we have $|C^{(j)}| \leq n/15$. Let $i > 1$ be the smallest index such that

$$\sum_{j=1}^{i} |C^{(j)} \cap L| > n/15,$$

and without loss of generality assume that $\sum_{j=1}^{i} |C^{(j)} \cap R| \leq \sum_{j=1}^{i} |C^{(j)} \cap L|$. Then, using $|C^{(j)}| \leq n/15$ for all $j \in [k]$, by minimality of i we have

$$\sum_{j=1}^{i} |C^{(j)} \cap R| \leq \sum_{j=1}^{i} |C^{(j)} \cap L| \leq 2n/15,$$

and therefore

$$\sum_{j=i+1}^{k} |C^{(j)} \cap R| \geq n/15.$$

Then by (P1) there exists an edge between $\sum_{j=1}^{i} |C^{(j)} \cap L|$ and $\sum_{j=i+1}^{k} |C^{(j)} \cap R|$, which contradicts the assumption that $C^{(1)}, \dots, C^{(k)}$ are the connected components of $G[L, R]$.

Let C be the vertices of the largest connected component of $G[L, R]$. As we have just showed, $|C| \geq n/15$. Let $y_1 \in C \cap L$ be an arbitrary vertex and let $g' = g - g(y_1) + f(y_1)$, i.e. the translation of g that agrees with f on y_1. Note that $L_g = L_{g'}$ and $R_g = R_{g'}$. Let $U := \{u \in [n] : f(u) = g'(u)\}$ be the set of vertices on which f agrees with g'. By the definition, we have $y_1 \in U$. We claim that the whole connected component C is contained in U. This is because for any vertex $x \in U$ and edge $xy \in E(G)$ of C, we also have $y \in U$: Suppose first that $x \in L_f \cap L_{g'}$ and $y \in R_f \cap R_{g'}$; the other case is analogous. Since x is in the left-half of f, y is in the right-half of f and because f realizes d we have $f(y) = f(x) + d(xy)$. Similarly, we obtain $g'(y) = g'(x) + d(xy)$. Using that $x \in U$, we conclude that $f(y) = g'(y)$, so $y \in U$.

Now $C \subseteq U$ implies $|U| \geq n/15$. If $U = [n]$ we are done. Otherwise (P2) can be applied and we take a vertex $v \in V(G) \setminus U$ which has two neighbors u_1, u_2 in U. Assume, by relabelling if necessary, that $f(u_1) = g'(u_1) < f(u_2) = g'(u_2)$. Since f realizes d, the f-value of v is determined by $f(u_1), f(u_2), d(u_1 v)$ and $d(u_2 v)$. Indeed, depending on whether $d(u_1 v)$, $d(u_2 v)$ or $|f(u_1) - f(u_2)|$ is the largest among the three, $f(v)$ is equal to $f(u_1) + d(u_1 v) = f(u_2) + d(u_2 v)$, $f(u_1) - d(u_1 v) = f(u_2) - d(u_2 v)$, or $f(u_1) + d(u_1 v) = f(u_2) - d(u_2 v)$, respectively. Since g' also realizes d, the value of $g'(v)$ is determined analogously. Finally, since $f(u_1) = g'(u_1)$ and $f(u_2) = g'(u_2)$, the values $f(v)$ and $g'(v)$ coming from the analogous formulas also agree. That means $v \in U$, contradicting $v \in V(G) \setminus U$. Thus, $U = V(G)$, and therefore, as $f = g'$, f and g are isometric. □

We now need the following simple property of random graphs.

Lemma 2.2. *For every $\varepsilon > 0$ there exists $C > 0$ such that if $m \geq Cn$, then $G \sim G(n, m)$ w.h.p. has the following property:*

(P3) For every disjoint $X, Y \subseteq V(G)$ of size $|X|, |Y| \geq \varepsilon n$, there exists an edge between X and Y in G.

Proof. For fixed X and Y, the probability that there is no edge between X and Y is

$$\binom{\binom{n}{2} - |X||Y|}{m} \Big/ \binom{\binom{n}{2}}{m} \leq e^{-|X||Y|m/n^2} < e^{-\varepsilon^2 C n}.$$

There are at most 2^{2n} ways to choose X and Y, thus, for $C > 2/\varepsilon^2$, w.h.p. this bad event does not happen for any such pair of sets. □

With Lemma 2.1 and Lemma 2.2 at hand, the proofs of Theorems 1.3 and 1.4 are straightforward.

Proof of Theorem 1.3. For the proof of Theorem 1.3 we check that w.h.p. both (P1) and (P2) hold for G_τ, so that the result follows directly by Lemma 2.1.

Let C be a constant given by Lemma 2.2 for $\varepsilon = 1/28$. It is well known [5] that w.h.p. $\tau \geq Cn := m$ (with C as given by Lemma 2.2). As G_m is uniformly distributed among all graphs with n vertices and exactly m edges, by Lemma 2.2 we have that w.h.p. (P3) holds in G_m. Since (P3) is monotone, it also holds in G_τ.

Property (P3) is straightforwardly stronger than (P1) and also implies (P2) in the case $n/15 \leq |U| \leq n/2$. Indeed, for the latter let $S \subseteq V(G) \setminus U$ be a subset of size εn. By (P3) we have

$$|N(S) \cap U| \geq |U| - \varepsilon n > |S|,$$

thus there exists a vertex in S with two neighbours in U. The remaining case $|U| > n/2$ of the property (P2) is proven to hold w.h.p., for example, in [11, Proposition 2.3]. □

Proof of Theorem 1.4. We can assume $\varepsilon > 0$ is sufficiently small. By Lemma 2.2, $G \sim G(n, m)$ w.h.h.p has the property (P3). This immediately implies (P1), thus to apply Lemma 2.1 we just need to find a large subset $V' \subseteq V(G)$ such that $G' = G[V']$ satisfies (P2). We define $V' := V(G) \setminus A$, where $A \subseteq V(G)$ is a largest subset such that $|A| \leq \varepsilon n$ and $|N(A)| \leq |A|$.

To check (P2) for a subset $U \subseteq V'$ of size $|V'|/15 \leq |U| < |V'| - \varepsilon n$ we consider a subset $S \subseteq V' \setminus U$ of size $\varepsilon n \leq |V' \setminus U|$. Applying (P3) we have

$$|N(S) \cap U)| \geq |U| - \varepsilon n > |S|,$$

which verifies that there is vertex in S with two G'-neighbors in U, for otherwise $|N(S) \cap U| \leq |U|$.

If $U \subseteq V'$ is of size $|U| \geq |V'| - \varepsilon n$, then we claim that for $S = V' \setminus U$ we have $|N(S) \setminus A| > |S|$, which in turn implies that some vertex of S has two neighbors in U. Otherwise we have

$$|N(A \cup S)| \leq |A| + |S| = |A \cup S|,$$

which implies $\varepsilon n < |A \cup S| \leq 2\varepsilon n$ by the maximality of A. We can then apply (P3) to obtain

$$|N(A \cup S)| \geq n - |A \cup S| - \varepsilon n > |A \cup S|,$$

a contradiction. □

3 Open problems

Random regular graphs. Following Benjamini and Tzalik [4], we studied the problem of global rigidity of random graphs in $\mathbb{R}$. A related natural question is, for which d is a random d-regular graph globally rigid with high probability? Using a result of Friedman [6] which shows that the second largest absolute eigenvalue of a random d-regular graph is, w.h.p., at most $2\sqrt{d-1} + \varepsilon$ for any $\varepsilon > 0$, together with the Expander Mixing Lemma and [1, Theorem 9.2.1], it is straightforward to verify that condition (P1) and (P2) hold for $d \geq 6$. This leaves the following problem open.

Problem 3.1. *Determine the smallest $d \geq 3$ for which a random d-regular graph with n vertices is globally rigid whp.*

Note that (P2) does not hold w.h.p. for a random 3-regular graph. Indeed, a random 3-regular graph w.h.p. contains a cycle C of length $O(\log n)$, thus (P2) fails for $U = V(G) \setminus V(C)$. This does not rule out the possibility that 3-random regular graphs are globally rigid, however a different strategy is likely needed.

Algorithmic problem. We note that in the case of a fixed function f, Benjamini and Tzalik [4], as well as Girão, Illingworth, Michel, Powierski, and Scott [8], also considered the algorithmic problem of finding f which realizes $d \colon E(G) \to \mathbb{R}^+$ when G is a random graph. They obtain algorithms with polynomial expected running time. In our setup, where we generate only one random graph to reconstruct any function f, our proof only provides a reconstruction algorithm with running time $O(2^n)$. We wonder whether this could be improved.

Problem 3.2. *Find an algorithm $\mathcal{A}$ with the following property: Let $G \sim G(n, p)$ for $p \gg \log n/n$. Then w.h.p. G is such that, for any injective $f \colon V(G) \to \mathbb{R}$, $\mathcal{A}(G, d_{f,G})$ finds in polynomial time (depending only on n) a function f' which realizes $d_{f,G}$.*

Higher dimensions. Finally, while one cannot hope for an extension of Theorem 1.3 to $\mathbb{R}^d$ for $d \geq 2$, it is conceivable that a statement of Theorem 1.4 is true for any $d \geq 2$. Even showing this for a given $f\colon [n] \to \mathbb{R}$ is an open problem, already suggested in [8], with some recent progress by Barnes, Petr, Portier, Shaw, and Sergeev [2]. Here we state the global rigidity version.

Problem 3.3. *Show that, for every integer $d \geq 2$ and $\varepsilon > 0$, there exists $C > 0$ such that the following holds. Let $G \sim G(n,p)$ for $p \geq C/n$. Then G w.h.p. has the following property: For every injective $f \in V(G) \to \mathbb{R}^d$ there exists a subset $V' \subseteq V(G)$ of size $|V'| \geq (1-\varepsilon)n$ such that f' is the only function (up to isometry) which realizes $d_{f',G'}$, where $G' = G[V']$ and $f' = f|_{V'}$.*

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