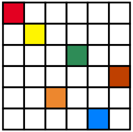
On the Ryser-Brualdi-Stein conjecture I



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Atlanta Lecture Series XXVII
in Combinatorics and Graph Theory

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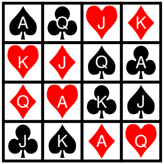
Latin square of order n	n by n grid filled with n symbols, where each symbol appears exactly once in each row and column

1	4	6	5	3	2
5	2	4	3	1	6
6	3	2	4	5	1
2	5	1	6	4	3
3	6	5	1	2	4
4	1	3	2	6	5

Full transversal	Set of n cells with different rows, columns and symbols

Euler: for which n is there a Latin square of order n which can be decomposed into n disjoint full transversals?



Euler: examples when $n \neq 2 \bmod 4$, and conjectured no examples exist if $n \equiv 2 \bmod 4$.

Tarry: showed no examples exist for $n = 6$ in 1901.

Conjecture (Euler, 1779)	If $n \equiv 2 \bmod 4$, there is no Latin square of order n with a decomposition into full transversals.

*ON THE FALSITY OF EULER'S CONJECTURE ABOUT THE NON-EXISTENCE OF TWO ORTHOGONAL LATIN SQUARES OF ORDER 4t + 2**

BY R. C. BOSE AND S. S. SHRIKHANDE

UNIVERSITY OF NORTH CAROLINA

Communicated by A. A. Albert, March 13, 1959

Bose, Parker, Shrikhande constructed examples for all $n \equiv 2 \bmod 4$ with $n \geq 10$.

0	6	5	4	9	8	7	1	2	3
7	1	0	6	5	9	8	2	3	4
8	7	2	1	0	6	9	3	4	5
9	8	7	3	2	1	0	4	5	6
1	9	8	7	4	3	2	5	6	0
3	2	9	8	7	5	4	6	0	1
5	4	3	9	8	7	6	0	1	2
2	3	4	5	6	0	1	7	8	9
4	5	6	0	1	2	3	8	9	7
6	0	1	2	3	4	5	9	7	8

Latin square of order n	n by n grid filled with n symbols, where each symbol appears exactly once in each row and column

Transversal	Set of cells with different rows, columns and symbols

Some Latin squares have **no** full transversal, e.g. the addition table for $\mathbb{Z}_{2k}$:

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Ryser-Brualdi-Stein Conjecture	Every Latin square of order n has a transversal with $n-1$ cells, and one with n cells if n is odd.

Ryser-Brualdi-Stein Conjecture	Every Latin square of order n has a transversal with $n-1$ cells, and one with n cells if n is odd.

Every Latin square of order n has a transversal with ...

... at least $n - \sqrt{n}$ cells.	Brouwer, De Vries and Wieringa (1978) Woolbright (1978)
... $n - O(\log^2 n)$ cells.	Shor (1982) Hatami-Shor (2008)
... $n - O\left(\frac{\log n}{\log \log n}\right)$ cells.	Keevash, Pokrovskiy, Sudakov and Yepremyan (2020)

Theorem (M., 23+)	There is some n_0 such that every Latin square of order $n \geq n_0$ has a transversal with $n-1$ cells.

Latin array of order n	n by n grid filled with symbols, where each symbol appears **at most** once in each row and column.

M., Pokrovskiy and Sudakov (2019) : if $\leq (1-\varepsilon)n$ symbols appear $\geq (1-\varepsilon)n$ times, there is a full transversal.

This allows:

Theorem (M., 23+)	There is some n_0 such that every Latin **array** of order $n \geq n_0$ has a transversal with $n-1$ cells.

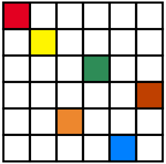
Conjecture (Akbari and Alipour) : Any Latin array of order n with $\geq \frac{n^2}{2}$ different symbols has a full transversal.

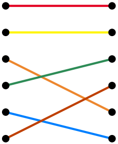
Keevash, Pokrovskiy, Sudakov and Yepremyan (2019) : For some C, $\geq \frac{Cn \log n}{\log \log n}$ different symbols forces a full transversal.

Theorem (M., 23+)	For some C, every Latin array of order n with $\geq Cn$ different symbols has a full transversal.

Ryser-Brualdi-Stein Conjecture	Every Latin square of order n has a transversal with $n-1$ cells, and one with n cells if n is odd.

Latin square of order n ⟷ Complete bipartite graph $K_{n,n}$, properly coloured with n colours



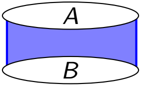


Transversal ⟷ Rainbow matching

Ryser-Brualdi-Stein Conjecture	If $K_{n,n}$ is optimally coloured, it contains a rainbow matching with $n-1$ edges, and with n edges if n is odd.

Extremal examples from groups

Let H be an n-element abelian group. Take two copies, A and B, of H, and for each $a \in A$ and $b \in B$, put an edge ab in G with colour $c(ab) = a + b \in H$:



If M is a perfect rainbow matching, then

$$\sum_{v \in H} v = \sum_{ab \in M} c(ab) = \sum_{ab \in M} (a + b) = 2 \sum_{v \in H} v,$$

so that $\sum_{v \in H} v = 0$.

In particular, if $H = \mathbb{Z}_{2m}$ then $\sum_{v \in H} v = \frac{(2m)(2m-1)}{2} = m \in H$, giving a contradiction.

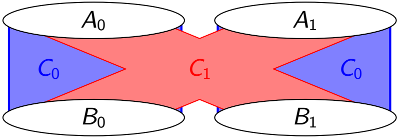
Which groups have a multiplication table (= a Latin square) with a full transversal?

This is known, due to the confirmation of the **Hall-Paige conjecture** by Bray, Wilcox and Evans in 2009, with more recent alternative proofs for large groups given by Eberhard, Manners and Mrazović, and Müyesser and Pokrovskiy.

Any more extremal examples?

We can generate more extremal colourings, using 'blow-up' constructions of group addition/multiplication tables. E.g., for $H = \mathbb{Z}_2$:

For $n = 2m$ with m odd, let $A_0, A_1, B_0, B_1, C_0, C_1$ be disjoint with size m, and properly colour edges between A_i and B_j with colours in C_{i+j}:



A similar calculation to before gives, if M is a rainbow matching,

$$m = m \cdot \sum_{v \in \mathbb{Z}_2} v = \sum_{e \in M} \sum_{v : e \in C_v} v = \sum_{w \in V(G)} \sum_{v : w \in A_v \cup B_v} v = 2m \cdot \sum_{v \in \mathbb{Z}_2} v = 2m,$$

a contradiction. (Or, in this case: any perfect matching has evenly many 'cross edges', so cannot be rainbow.)

This does give many extremal examples, but a uniformly random Latin square does have a full transversal with high probability (Kwan, 2020).

Ryser-Brualdi-Stein Conjecture	If $K_{n,n}$ is optimally coloured, it contains a rainbow matching with $n-1$ edges, and with n edges if n is odd.

Overall strategy

1. Study the colouring and determine some algebraic properties.
2. Use these properties to construct a large rainbow matching.

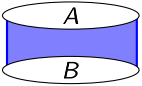
Rest of today:
- Part 1

Tomorrow:
- Recap
- Part 2 (under a simplifying assumption avoiding Part 1)
- Open questions

Sneak peak: Part 2 uses the semi-random method and absorption, along with a new 'addition structure', for the construction.

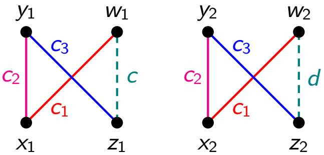
Part 1: Algebraic properties of colourings

G: n by n complete bipartite graph, properly coloured with n colours.



If the colouring of G arises from an abelian group H, then the following holds.

Property P. For all (distinct) $c_1, c_2, c_3 \in H$ and $w_1, w_2 \in H$:



$\Longrightarrow c = d$

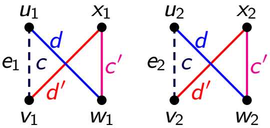
- Indeed, labelling the paths $w_1x_1y_1z_1$ and $w_2x_2y_2z_2$, we have

$$c = w_1 + z_1 = (w_1 + x_1) - (x_1 + y_1) + (y_1 + z_1) = c_1 - c_2 + c_3 = d.$$

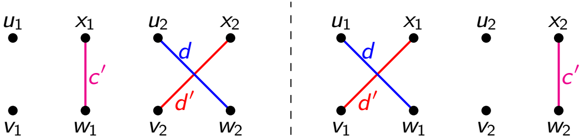
- (Slightly roughly) if **Property P** holds, then the colouring comes from some group.

[Ignorable slide]

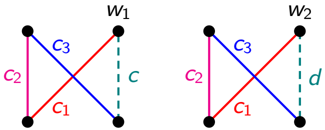
If we have **P** here:



then we can create an edge-switcher to switch between using $\{u_2, v_2\}$ and $\{u_1, v_1\}$ (where these are vertex sets of edges of the same colour):



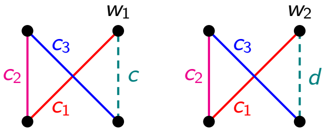
Property P. For all (distinct) $c_1, c_2, c_3 \in H$ and $w_1, w_2 \in H$:



$\implies c = d$

In general, **Property P** does not hold, and instead we look for colour classes $C_1, \dots, C_r \subset C(G)$ (for some r) for which **Property P'** (approximately) holds:

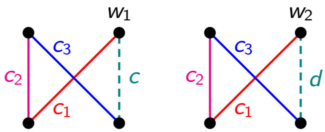
Property P'. For all (distinct) $c_1, c_2, c_3 \in C(G)$ and $w_1, w_2 \in V(G)$:



$\implies$ $\exists i$ with $c, d \in C_i$

Of course, this is always true if $C_1 = C(G)$, so the aim is to do this using as small sets C_i as possible, and balance this with proving a property for colours in each class C_i.

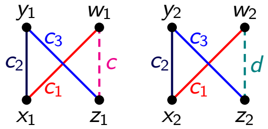
Property P'. For all (distinct) $c_1, c_2, c_3 \in C(G)$ and $w_1, w_2 \in V(G)$:



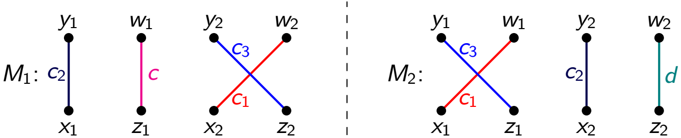
$\Longrightarrow$ $\exists i$ with $c, d \in C_i$

What properties should we have for a colour class C_i?

For any pair of colour $c, d \in C_i$ we want to be able to consider them to be equivalent in our subsequent constructions. Suppose we have the following:



Then, we can use this as a (c, d)-colour-switcher:



Some examples

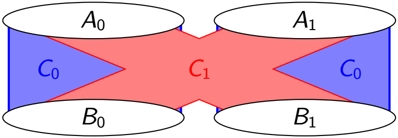
1. If the colouring comes from the group addition table of an abelian group G, the colour classes are $\{v\}$, $v \in G$.

- **Property P'** follows from **Property P**.
- The switching property is trivial.

2. If we take a uniformly random optimal colouring of $K_{n,n}$ then (whp) there is one colour class: $C_1 = C(G)$.

- **Property P'** is then trivial.
- Harder: for any pair of colours c, d we expect some c, d-colour-switchers.

3. For $n = 2m$ with m odd, let $A_0, A_1, B_0, B_1, C_0, C_1$ be disjoint with size m. **Randomly**, properly colour edges between A_i and B_j with colours in C_{i+j} (adding in $\mathbb{Z}_2$):



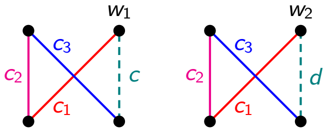
We use colour classes C_0 and C_1.

- **Property P'** follows from the construction
- The colour switching property follows (again) from the randomness.

More generally...

- Proof considers the complete auxiliary graph K with vertex set $C(G)$, and each edge weighted by the number of short c, d-colour-switchers in G.
- Partitions most of K into well-connected subgraphs with similar edge weights (via sublinear expansion), and takes their vertex sets in K as colour classes C_i.

Property P'. For all (distinct) $c_1, c_2, c_3 \in C(G)$ and $w_1, w_2 \in V(G)$:



$\Longrightarrow$ $\exists i$ with $c, d \in C_i$

- **Property P'** will follow (approximately) as the subgraphs cover almost all of K.
- The colour switching property will follow as the connectedness of $K[C_i]$ allows short colour-switchers to be chained together into a longer c, d-colour-switcher, for any $c, d \in C_i$.

Summary

Ryser-Brualdi-Stein Conjecture	If $K_{n,n}$ is optimally coloured, it contains a rainbow matching with $n-1$ edges, and with n edges if n is odd.

- Numerous extremal colourings exist, each with some underlying algebraic properties.

Overall strategy for (n-1)-edge rainbow matchings

1. Study the colouring and determine some algebraic properties.
2. Use these properties to construct a large rainbow matching.

Property P is a graph-theoretic equivalent to having a colouring from an abelian group.

Most of the work in Part 1 goes into finding colour classes for which the adjusted **Property P'** (approximately) holds, and any pair of colours from the same class have colour switchers, which allow the constructions for Part 2 to work in any colouring.

For simplicity tomorrow I will assume that **Property P** holds, and discuss Part 2 ...

Thank you!