ASYMPTOTIC SIZE RAMSEY RESULTS FOR BIPARTITE GRAPHS*

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Abstract. We show that $\lim_{n\to\infty} \hat{r}(F_{1,n},\ldots,F_{q,n},F_{p+1},\ldots,F_r)/n$ exists, where the bipartite graphs $F_{q+1},\ldots,F_r$ do not depend on n while, for $1\le i\le q$, $F_{i,n}$ is obtained from some bipartite graph F_i with parts $V_1\cup V_2=V(F_i)$ by duplicating each vertex $v\in V_2$ $(c_v+o(1))n$ times for some real $c_v>0$.

In fact, the limit is the minimum of a certain mixed integer program. Using the Farkas Lemma we show how to compute it when each forbidden graph is a complete bipartite graph, in particular answering the question of Erdős, Faudree, Rousseau and Schelp (1978) who asked for the asymptotics of $\hat{r}(K_{s,n},K_{s,n})$ for fixed s and large n. Also, we prove (for all sufficiently large n) the conjecture of Faudree, Rousseau and Sheehan (1983) that $\hat{r}(K_{2,n},K_{2,n})=18n-15$.

1. Introduction.

Let $(F_1,\ldots,F_r)$ be an r-tuple of graphs which are called *forbidden*. We say that a graph G *arrows* $(F_1,\ldots,F_r)$ if for any r-colouring of $E(G)$, the edge set of G, there is a copy of F_i of colour i for some $i\in[r]:=\{1,\ldots,r\}$. We denote this *arrowing property* by $G\to(F_1,\ldots,F_r)$.

The (ordinary) *Ramsey number* asks for the minimum order of such G. Here, however, we deal exclusively with the *size Ramsey number*

$$\hat{r}(F_1,\ldots,F_r)=\min\{e(G): G\to(F_1,\ldots,F_r)\}$$

which is the smallest number of edges that an arrowing graph can have.

Size Ramsey numbers seem hard to compute, even for simple forbidden graphs. For example, the old conjecture of Erdős [6] that $\hat{r}(K_{1,n},K_3)=3n(n+1)/2$ has only recently been disproved in [16], where it is shown that $\hat{r}(K_{1,n},F)=(1+o(1))n^2$ for any fixed 3-chromatic graph F. (Here, $K_{m,n}$ is the complete bipartite graph with parts of sizes m and n; K_n is the complete graph of order n.)

This research initiated as an attempt to find the asymptotics of $\hat{r}(K_{1,n},F)$ for a fixed graph F. The case $\chi(F)\ge 4$ is treated in [14] (and [16] deals with $\chi(F)=3$). What can be said if F is a bipartite graph?

Faudree, Rousseau and Sheehan [10] proved that

$$\hat{r}(K_{1,n},K_{2,m})=4n+2m-4$$

for every $m\ge 9$ if n is sufficiently large (depending on m) and stated that their method shows that $\hat{r}(K_{1,n},K_{2,2})=4n$, $n\ge 3$. They also observed that $K_{s,2n}$ arrows the pair $(K_{1,n},C_{2s})$ for $n\ge s$, where C_{2s} is the cycle of order $2s$; hence $\hat{r}(K_{1,n},C_{2s})\le 2sn$ then.

Let P_s be the path with s vertices. Lortz and Mengersen [13] showed that $K_{k,2n-1}\to(K_{1,n},P_{2k+1})$ and $K_k+\overline{K}_{2n-k-1}\to(K_{1,n},P_{2k})$ and conjectured that

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this is sharp for any $s \geq 4$ provided n is sufficiently large, that is,

$$\hat{r}(K_{1,n}, P_s) = \begin{cases} 2kn - k, & \text{if } s = 2k+1, \\ 2kn - k(k+3)/2, & \text{if } s = 2k, \end{cases} \qquad n \geq n_0(s). \tag{1.1}$$

The conjecture was proved for $4 \leq s \leq 7$ in [13].

Size Ramsey numbers $\hat{r}(F_1, F_2)$ for bipartite graphs F_1 and F_2 are also studied in [8, 5, 2, 3, 7, 9, 12, 11] for example.

It is not hard to see that, for fixed $s_1, \dots, s_r \in \mathbb{N}$ and $t_1, \dots, t_r \in \mathbb{R}_{>0}$, we have

$$\hat{r}(K_{s_1, \lfloor t_1 n \rfloor}, \dots, K_{s_r, \lfloor t_r n \rfloor}) = O(n). \tag{1.2}$$

This follows, for example, by assuming that $s_1 = \dots = s_r = s$, $t_1 = \dots = t_r = t$ and considering K_{v_1,v_2}, where $v_1 = (s-1)r + 1$ and $v_2 = \lceil rtn\binom{v_1}{s} \rceil$. The latter graph has the required arrowing property. Indeed, for any r-colouring, each vertex of V_2 is incident to at least s edges of the same colour; hence there are at least v_2 monochromatic $K_{s,1}$-subgraphs and some $S \in \binom{V_1}{s}$ appears in at least $v_2/\binom{v_1}{s} \geq rtn$ such subgraphs of which at least tn have the same colour.

Here we will show that the limit $\lim_{n\to\infty} \hat{r}(F_{1,n}, \dots, F_{r,n})/n$ exists if each forbidden graph is either a fixed bipartite graph or a subgraph of $K_{s,\lfloor tn \rfloor}$ which 'dilates' uniformly with n (the precise definition will be given in Section 2). In particular, $\hat{r}(K_{1,n}, F)/n$ tends to a limit for any fixed bipartite graph F.

The limit value can in fact be obtained as the minimum of a certain mixed integer program (which does depend on n). We have been able to solve the MIP when each $F_{i,n}$ is a complete bipartite graph. In particular, we answer the question of Erdős, Faudree, Rousseau and Schelp [8, Problem B] who asked for the asymptotics of $\hat{r}(K_{s,n}, K_{s,n})$. Working harder on the case $s = 2$ we prove (for all sufficiently large n, $n \geq n_0$) the conjecture of Faudree, Rousseau and Sheehan [10, Conjecture 15] that

$$\hat{r}(K_{2,n}, K_{2,n}) = 18n - 15, \tag{1.3}$$

where the upper bound is obtained by considering $K_{3,6n-5} \to (K_{2,n}, K_{2,n})$. The identity (1.3) is not true for all n: for example, it is stated in [10] that $\hat{r}(K_{2,2}, K_{2,2}) = 15$. The upper bound follows from $K_6 \to (K_{2,2}, K_{2,2})$, which is easy to verify. Our method could produce a concrete value for n_0 with extra tedious calculations, but this would probably be rather large.

Unfortunately, our MIP is not well suited for practical calculations and we were not able to compute the asymptotics for any other non-trivial forbidden graphs; in particular, we had no progress on (1.1). But we hope that the introduced method will produce more results: although the MIP is hard to solve, it may be possible that, for example, some manageable relaxation of it gives good lower or upper bounds.

Our method does not work if we allow both vertex classes of forbidden graphs to grow with n. In these settings, in fact, we do not know the asymptotics even in the simplest cases. For example, the best known bounds on $r = \hat{r}(K_{n,n}, K_{n,n})$ seem to be $r < \frac{3}{2}n^3 2^n$ for $n \geq 6$ (Erdős, Faudree, Rousseau and Schelp [8]) and $r > \frac{1}{60}n^2 2^n$ for $n \geq n_0$ (Erdős and Rousseau [9]).

2. Main Ideas and Definitions. Let us briefly describe the main ideas behind our approach and how they came into existence. As an illustration, suppose we want to prove that $\hat{r}(K_{2,n}, K_{2,n}) \geq (18 + o(1))n$. Let n be large and let $G \to (K_{2,n}, K_{2,n})$

be any graph with $e(G) \le (18 + o(1))n$. We try to get as much information about the structure of G as possible.

Let $L = \{x \in V(G) : d(x) \ge n\}$. Clearly, $|L| \le 18$. As no edge disjoint from L lies inside a $K_{2,n}$-subgraph of G, we can harmlessly remove all such edges from G, that is, we can assume that $V(G) \setminus L$ is an independent set in G.

Also, if we remove all edges within L, the arrowing property is only slightly impaired: the obtained graph G' arrows $(K_{2,n'}, K_{2,n'})$, where we can take $n' = n - 16$ (or even larger). So, replacing G by G' and n by n', we can assume that $G \subset K_{18,m}$ for some $m = m(n)$.

Also, we can assume that every vertex of L has degree at least $2n - 1$. (This is not crucial here but this illustrates Lemma 3.1.) Indeed, if we remove any $x \in L$ of degree at most $2n - 2$, then the obtained graph G' arrows $(K_{2,n-1}, K_{2,n-1})$: any $(K_{2,n-1}, K_{2,n-1})$-free colouring of G' extends to a $(K_{2,n}, K_{2,n})$-free colouring of G by colouring the remaining edges without a monochromatic $K_{1,n}$ centred at x.

Thus, we can assume that $G \subset K_{9,m}$. How can we economically describe such a graph? This brings us to new definitions.

Let F be a bipartite graph. We assume that bipartite graphs come equipped with a fixed bipartition $V(F) = V_1(F) \cup V_2(F)$, although graph embeddings need not preserve it. We denote $v_i(F) = |V_i(F)|$, $i = 1, 2$; thus $v(F) = v_1(F) + v_2(F)$. Define,

$$F^A = \{v \in V_2(F) : \Gamma_F(v) = A\}, \quad A \subset V_1(F),$$

where $\Gamma_F(v)$ denotes the neighbourhood of v in F. (We will write $\Gamma(v)$, etc., when the encompassing graph F is clear from the context.) Clearly, in order to determine F (up to an isomorphism) it is enough to know $V_1(F)$ and $|F^A|$ for all $A \in 2^{V_1(F)}$.

Now, instead of dealing with $G \to (K_{2,n}, K_{2,n})$ we prefer to work with the numbers $|G^A|$. As $e(G) = O(n)$, we can let $n \to \infty$ over some sequence so that $|G^A|/n$ tends to a limit g_A for each $A \in 2^L$. The, how we call it, 'weight' $\mathbf{g} = (g_A)_{A \in 2^L}$ cannot be arbitrary: the fact that $G \to (K_{2,n}, K_{2,n})$ imposes some restrictions on $\mathbf{g}$. The question arises whether we can rephrase the arrowing property for weights without appealing to the original graphs. This requires rewriting the notions of a subgraph, colouring, etc. For the sake of generality, one would also wish to allow constant (i.e., not depending on n) forbidden subgraphs, which prompts one to define the mixed relation '$F \subset \mathbf{g}$' as well, where F is a graph and $\mathbf{g}$ is a weight. This is the first part of the programme, which culminates in Theorem 3.3, where it is shown that the 'weight size Ramsey number' gives indeed the asymptotics of the ordinary number. However, the second part, to calculate the weight size Ramsey number, is not an easy task and we are able to carry it out for complete bipartite graphs only.

Let us give formal definitions. A *weight* $\mathbf{f}$ on a set $V(\mathbf{f})$ is a sequence $(f_A)_{A \in 2^{V(\mathbf{f})}}$ of non-negative reals. A bipartite graph F *agrees* with $\mathbf{f}$ if $V_1(F) = V(\mathbf{f})$ and $F^A = \emptyset$ if and only if $f_A = 0$, $A \in 2^{V(\mathbf{f})}$. A sequence of bipartite graphs $(F_n)_{n \in \mathbb{N}}$ is a *dilatation* of $\mathbf{f}$ (or *dilates* $\mathbf{f}$) if each F_n agrees with $\mathbf{f}$ and

$$|F_n^A| = f_A n + o(n), \quad \text{for all } A \in 2^{V(\mathbf{f})}.$$

(Of course, the latter condition is automatically true for all $A \in 2^{V(\mathbf{f})}$ with $f_A = 0$.) Clearly, $e(F_n) = (e(\mathbf{f}) + o(1))n$, where $e(\mathbf{f}) = \sum_{A \in 2^{V(\mathbf{f})}} f_A\, |A|$, so we call $e(\mathbf{f})$ the *size* of $\mathbf{f}$. Also, the *order* of $\mathbf{f}$ is $v(\mathbf{f}) = |V(\mathbf{f})|$ and the *degree* of $x \in V(\mathbf{f})$ is

$$d(x) = \sum_{\substack{A \in 2^{V(\mathbf{f})} \\ A \ni x}} f_A.$$

Clearly, $e(\mathbf{f}) = \sum_{x\in V(\mathbf{f})} d(x)$.

For example, given $t \in \mathbb{R}_{>0}$, the sequence $(K_{s,\lceil tn\rceil})_{n\in\mathbb{N}}$ is the dilatation of $\mathbf{k}_{s,t}$, where the symbol $\mathbf{k}_{s,t}$ will be reserved for the weight on $[s]$ which has value t on $[s]$ and zero otherwise. (We assume that $V_1(K_{s,m}) = [s]$.) It is not hard to see that any sequence of bipartite graphs described in the abstract is in fact a dilatation of some weight.

We write $F \subset \mathbf{f}$ if for some bipartition $V(F) = V_1(F) \cup V_2(F)$ there is an injection $h : V_1(F) \to V(\mathbf{f})$ such that for any $A \subset V_1(F)$ dominated by a vertex of $V_2(F)$ there is $B \subset V(\mathbf{f})$ with $B \supset h(A)$ and $f_B > 0$. This notation is motivated by the following easy lemma. In fact, we will implicitly prove a sharper version during the proof of Theorem 3.3, so we give no proof here.

LEMMA 2.1. *Let $(F_n)_{n\in\mathbb{N}}$ be a dilatation of $\mathbf{f}$. If $F \subset \mathbf{f}$, then F is a subgraph of F_n for all sufficiently large n. Otherwise, which is denoted by $F \not\subset \mathbf{f}$, no F_n contains F.* □

Next, we define the '$\subset$'-relation between two weights $\mathbf{f}$ and $\mathbf{g}$. Assume that $v(\mathbf{f}) \le v(\mathbf{g})$ by adding new vertices to $V(\mathbf{g})$ and letting $\mathbf{g}$ be zero on all new sets. We write $\mathbf{f} \subset \mathbf{g}$ if there is an injection $h : V(\mathbf{f}) \to V(\mathbf{g})$ and numbers $(w_{AB} \ge 0)_{A\in 2^{V(\mathbf{f})},\, B\in 2^{V(\mathbf{g})}}$ such that

$$\begin{aligned}
\forall A \in 2^{V(\mathbf{f})},\ \forall B \in 2^{V(\mathbf{g})} \quad & h(A) \not\subset B \Rightarrow w_{AB} = 0,\\
\forall A \in 2^{V(\mathbf{f})} \quad & \sum_{\substack{B\in 2^{V(\mathbf{g})}\\ B\supset h(A)}} w_{AB} \ge f_A,\\
\forall B \in 2^{V(\mathbf{g})} \quad & \sum_{\substack{A\in 2^{V(\mathbf{f})}\\ h(A)\subset B}} w_{AB} \le g_B.
\end{aligned}$$

This definition is a bit difficult to comprehend. In a sense, it corresponds to a graph embedding $F \subset G$ preserving the $V_1 \cup V_2$-partition: h embeds $V_1(F)$ into $V_1(G)$ and $w_{A,B}$ says how much of $F^A \subset V_2(F)$ is mapped into G^B. The motivation comes from the following lemma which, like Lemma 2.1, is stated without a proof.

LEMMA 2.2. *Let $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ be dilatations of $\mathbf{f}$ and $\mathbf{g}$ respectively. Then $\mathbf{f} \subset \mathbf{g}$ implies that for any $\epsilon > 0$ there is n_0 such that $F_n \subset G_m$ for any $n \ge n_0$ and $m \ge (1+\epsilon)n$. Otherwise, which is denoted by $\mathbf{f} \not\subset \mathbf{g}$, there is $\epsilon > 0$ and n_0 such that $F_n \not\subset G_m$ for any $n \ge n_0$ and $m \le (1+\epsilon)n$.* □

The weight $\subset$-relation enjoys many properties of the graph one. For example, $d(x) \le d(h(x))$ for any $x \in V(\mathbf{f})$:

$$d(x) = \sum_{\substack{A\in 2^{V(\mathbf{f})}\\ A\ni x}} f_A \le \sum_{\substack{A\in 2^{V(\mathbf{f})}\\ A\ni x}} \sum_{\substack{B\in 2^{V(\mathbf{g})}\\ B\supset h(A)}} w_{A,B} \le \sum_{\substack{B\in 2^{V(\mathbf{g})}\\ B\ni h(x)}} \sum_{\substack{A\in 2^{V(\mathbf{f})}\\ h(A)\subset B}} w_{A,B} \le \sum_{\substack{B\in 2^{V(\mathbf{g})}\\ B\ni h(x)}} g_B = d(h(x)). \tag{2.1}$$

An *r-colouring* $\mathbf{c}$ of $\mathbf{g}$ is a sequence $(c_{A_1,\dots,A_r})$ of non-negative reals indexed by r-tuples of disjoint subsets of $V(\mathbf{g})$ such that

$$\sum_{A_1\cup\cdots\cup A_r = A} c_{A_1,\dots,A_r} > g_A, \quad \text{for all } A \in 2^{V(\mathbf{g})}. \tag{2.2}$$

The *i-th colour subweight* $\mathbf{c}_i$ is defined by $V(\mathbf{c}_i) = V(\mathbf{g})$ and

$$c_{i,A} = \sum_{\substack{A_1,\dots,A_r\\ A_i = A}} c_{A_1,\dots,A_r}, \quad A \in 2^{V(\mathbf{g})}. \tag{2.3}$$

The analogy: to define an r-colouring of G, it is enough to define, for all disjoint $A_1,\dots,A_r \subset V_1(G)$, how many vertices of $G^{A_1\cup\dots\cup A_r}$ are connected, for all $i\in[r]$, by colour i precisely to A_i. We put the strict inequality in (2.2) so that Lemma 3.2 is true.

3. Existence of Limit. Let $r \ge q \ge 1$. Consider a sequence $\mathbf{F} = (\mathbf{F}_1,\dots,\mathbf{F}_r)$, where $\mathbf{F}_i = \mathbf{f}_i$ is a weight for $i\in[q]$ and $\mathbf{F}_i = F_i$ is a bipartite graph for $i \in [q+1,r]$. Assume that $\mathbf{F}_i$ does not have an *isolated vertex* (that is, $x\in V(\mathbf{F}_i)$ with $d(x)=0$), $i\in[r]$. We say that a weight $\mathbf{g}$ *arrows* $\mathbf{F}$ (denoted by $\mathbf{g}\to\mathbf{F}$) if for any r-colouring $\mathbf{c}$ of $\mathbf{g}$ we have $\mathbf{F}_i \subset \mathbf{c}_i$ for some $i\in[r]$. Define

$$\hat{r}(\mathbf{F}) = \inf\{e(\mathbf{g}) : \mathbf{g}\to\mathbf{F}\}. \tag{3.1}$$

The definition (3.1) imitates that of the size Ramsey number and we will show that these are very closely related indeed. However, we need a few more preliminaries.

Observe that $\hat{r}(\mathbf{F}) < \infty$ by considering $\mathbf{k}_{a,b}$ which arrows $\mathbf{F}$ if, for example, $a = 1+\sum_{i=1}^r (v(\mathbf{F}_i)-1)$ and b is sufficiently large, cf. (1.2). Let l be an integer greater than $\hat{r}(\mathbf{F})/d_0$, where $d_0 = \sum_{i=1}^q d_i$ and

$$d_i = \min\{d_{\mathbf{f}_i}(x) : x\in V(\mathbf{f}_i)\} > 0,\quad i\in[q].$$

LEMMA 3.1. *Let $\mathbf{g}\to\mathbf{F}$ have no isolated vertices. If $d_{\mathbf{g}}(x) < d_0$ for some $x\in V(\mathbf{g})$ or if $v(\mathbf{g}) > l$, then there is $\mathbf{g}'\to\mathbf{F}$ with $e(\mathbf{g}') < e(\mathbf{g})$ and $v(\mathbf{g}') < v(\mathbf{g})$.*

It follows that $\hat{r}(\mathbf{F}) = \hat{r}_l(\mathbf{F})$, where $r_l(\mathbf{F}) = \min\{e(\mathbf{g}) : \mathbf{g}\to\mathbf{F},\ v(\mathbf{g})\le l\}$.

Proof. Let $d(x) < d_0$. Choose $\delta > 0$ with $\delta + d_i d(x)/d_0 < d_i$ for any $i\in[q]$. Define the weight $\mathbf{g}'$ on $V(\mathbf{g})\setminus\{x\}$ by $g'_A = g_A + g_{A\cup\{x\}}$, $A\in 2^{V(\mathbf{g}')}$. Clearly, $e(\mathbf{g}') = e(\mathbf{g}) - d(x) < e(\mathbf{g})$.

We claim that $\mathbf{g}'$ arrows $\mathbf{F}$. Suppose that this is not true and let $\mathbf{c}'$ be an $\mathbf{F}$-free r-colouring of $\mathbf{g}'$. We can assume that

$$\sum_{A_1\cup\dots\cup A_r = A} c'_{A_1,\dots,A_r} \le g'_A + \delta/2^{v(\mathbf{g}')},\quad \text{for any } A\in 2^{V(\mathbf{g}')}.$$

Define $\mathbf{c}$ by

$$c_{A_1,\dots,A_r} = \begin{cases} \frac{\lambda_{A\setminus\{x\}} d_i}{d_0}\cdot c'_{A_1,\dots,A_{i-1},A_i\setminus\{x\},A_{i+1},\dots,A_r}, & x\in A_i,\ i\in[q],\\ 0, & x\in A_{q+1}\cup\ldots\cup A_r,\\ (1-\lambda_A)\cdot c'_{A_1,\dots,A_r}, & x\notin A,\end{cases}$$

where we denote $A = A_1\cup\dots\cup A_r$, $\lambda_A = g_{A\cup\{x\}}/g'_A$ if $g'_A > 0$, and $\lambda_A = 1/2$ if $g'_A = 0$. The reader can check that $\mathbf{c}$ is an r-colouring of $\mathbf{g}$.

By the assumption on $\mathbf{g}$, we have $\mathbf{F}_i\subset\mathbf{c}_i$ for some $i\in[r]$. But this embedding cannot use x because for $i\in[q+1,r]$ we have $d_{\mathbf{c}_i}(x) = 0$ while for $i\in[q]$

$$\begin{aligned} d_{\mathbf{c}_i}(x) &= \sum_{A_1,\dots,A_r\subset V(\mathbf{g}')} c_{A_1,\dots,A_{i-1},A_i\cup\{x\},A_{i+1},\dots,A_r} = \sum_{A\in 2^{V(\mathbf{g}')}} \frac{\lambda_A d_i}{d_0} \sum_{A_1\cup\dots\cup A_r = A} c'_{A_1,\dots,A_r}\\ &\le \sum_{A\in 2^{V(\mathbf{g}')}} \frac{\lambda_A d_i}{d_0}(g'_A + \delta/2^{v(\mathbf{g}')}) \le \frac{d_i\delta}{d_0} + \frac{d_i}{d_0}\sum_{A\in 2^{V(\mathbf{g}')}} g_{A\cup\{x\}} \le \delta + d_i\frac{d(x)}{d_0} < d_i \end{aligned}$$

is too small, see (2.1). But $c_{i,A}\le c'_{i,A}$ for $A\in 2^{V(\mathbf{g}')}$; hence, $\mathbf{F}_i\subset\mathbf{c}'_i$, which is the desired contradiction proving the first claim.

Let $v(\mathbf{g}) > l$. If $e(\mathbf{g}) \geq \hat{r}(\mathbf{F}) + d_0$, replace $\mathbf{g}$ by any other arrowing weight with $e(\mathbf{g}) < \hat{r}(\mathbf{F}) + d_0$. As $e(\mathbf{g})/(l+1) < d_0$, we can eventually ensure that $v(\mathbf{g}) \leq l$ by iterating the first claim. $\square$

Hence, to compute $\hat{r}(\mathbf{F})$ it is enough to consider $\mathbf{F}$-arrowing weights on $L = [l]$ only.

LEMMA 3.2. *There exists* $\mathbf{g} \to \mathbf{F}$ *with* $V(\mathbf{g}) \subset L$ *and* $e(\mathbf{g}) = \hat{r}(\mathbf{F})$. *(And we call such a weight* extremal.*)*

Proof. Choose $\mathbf{g}_n \to \mathbf{F}$ with $V(\mathbf{g}_n) \subset L$, $n \in \mathbb{N}$, such that $e(\mathbf{g}_n)$ approaches $\hat{r}(\mathbf{F})$. By choosing a subsequence, assume that $V(\mathbf{g}_n)$ is constant and $g_A = \lim_{n\to\infty} g_{n,A}$ exists for each $A \in 2^L$. Clearly, $e(\mathbf{g}) = \hat{r}(\mathbf{F})$ so it remains to show that $\mathbf{g} \to \mathbf{F}$.

Let $\mathbf{c}$ be an r-colouring of $\mathbf{g}$. Let δ be the smallest slack in inequalities (2.2). Choose sufficiently large n so that $|g_{n,A} - g_A| < \delta$ for all $A \in 2^L$. We have

$$\sum_{A_1\cup\cdots\cup A_r = A} g_{A_1,\ldots,A_r} \geq g_A + \delta > g_{n,A}, \quad A \in 2^L,$$

that is, $\mathbf{c}$ is a colouring of $\mathbf{g}_n$ as well. Hence, $\mathbf{F}_i \subset \mathbf{c}_i$ for some i, as required. $\square$

Now we are ready to prove our general theorem. The proof essentially takes care of itself. We just exploit the parallels between weights and graphs, which, unfortunately, requires messing around with various constants.

THEOREM 3.3. *Let* $(F_{i,n})_{n\in\mathbb{N}}$ *be a dilatation of* $\mathbf{f}_i$, $i \in [q]$, *and let* F_i *be a fixed bipartite graph,* $i \in [q+1, r]$. *Then, for all sufficiently large* n,

$$\hat{r}(\mathbf{F})n - M(1+f_0) \leq \hat{r}(F_{1,n},\ldots,F_{q,n},F_{q+1},\ldots,F_r) \leq \hat{r}(\mathbf{F})n + M(1+f_0), \tag{3.2}$$

where $f_0 = \max\{\,||F_{i,n}^A| - f_{i,A}n|\, : i \in [q], A \in V(\mathbf{f}_i)\}$ *and* $M = M(\mathbf{F})$ *is some constant.*

In particular, the limit $\lim_{n\to\infty} \hat{r}(F_{1,n},\ldots,F_{q,n},F_{q+1},\ldots,F_r)/n$ *exists.*

Proof. Let $v_0 = \max\{v(F_i) : i \in [r]\}$, $m_1 = 2^{v_0}(f_0+1)$, and $m_2 = r^l m_1 + 1$, where, as before, $l > \hat{r}(\mathbf{F})/d_0$.

We prove that

$$\hat{r}(F_1,\ldots,F_r) \leq \hat{r}(\mathbf{F})n + 2^l l(m_2+1), \quad n \geq 1. \tag{3.3}$$

By Lemma 3.2 choose an extremal weight $\mathbf{g}$ on L. Define a bipartite graph G as follows. Choose disjoint from each other (and from L) sets G^A with $|G^A| = \lceil g_A n + m_2 \rceil$, $A \in 2^L$. Let $V(G) = L \cup (\cup_{A\in 2^L} G^A)$. In G we connect $x \in L$ to everything in G^A if $x \in A$. These are all the edges. Clearly,

$$e(G) = \sum_{A\in 2^L} |G^A|\,|A| \leq 2^l l(m_2+1) + \sum_{A\in 2^L} g_A n\,|A| \leq 2^l l(m_2+1) + \hat{r}(\mathbf{F})n,$$

as required. Hence, it is enough to show that G has the arrowing property.

Consider any r-colouring $c : E(G) \to [r]$. For every r-tuple of disjoint sets $B_1,\ldots,B_r \subset L$, let

$$C_{B_1,\ldots,B_r} = \{y \in G^B : \forall i \in [r]\ \forall x \in B_i \ \ c(\{x,y\}) = i\},$$
$$c_{B_1,\ldots,B_r} = \begin{cases} (|C_{B_1,\ldots,B_r}| - m_1)/n, & \text{if } |C_{B_1,\ldots,B_r}| \geq m_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $B = B_1 \cup \cdots \cup B_r$. In any case, $nc_{B_1,\ldots,B_r} \geq |C_{B_1,\ldots,B_r}| - m_1$; hence, for every $B \in 2^L$ we have

$$n \sum_{B_1\cup\cdots\cup B_r = B} c_{B_1,\ldots,B_r} \geq -r^{|B|}m_1 + \sum_{B_1\cup\cdots\cup B_r = B} |C_{B_1,\ldots,B_r}| \geq -r^l m_1 + |G^B| > ng_B,$$

that is, $\mathbf{c}$ is an r-colouring of $\mathbf{g}$. Hence, $\mathbf{F}_i \subset \mathbf{c}_i$ for some $i \in [r]$. Now we show that G contains a forbidden subgraph in the ith colour.

Suppose that $i \in [q]$. By the definition, we find appropriate $h : V(\mathbf{f}_i) \to L$ and $\mathbf{w}$. We aim at proving that $F_{i,n} \subset G_i$, where $G_i \subset G$ is the colour-i subgraph. Partition $F_{i,n}^A = \cup_{B \supset h(A)} W_{A,B}$ so that $W_{A,B} = \emptyset$ if $w_{A,B} = 0$ and $|W_{A,B}| \le \lfloor w_{A,B} n + f_0 + 1 \rfloor$, $A \in 2^{V(\mathbf{f}_i)}$, $B \in 2^L$. This is possible for any A: if $w_{A,B} = 0$ for all $B \in 2^L$ with $h(A) \subset B$, then $f_{i,A} = 0$ and $F_{i,n}^A = \emptyset$; if $w_{A,B} > 0$ for at least one B, then

$$\sum_{\substack{B \in 2^L \\ w_{A,B} > 0}} (w_{A,B} n + f_0) \ge f_0 + n \sum_{\substack{B \in 2^L \\ w_{A,B} > 0}} w_{A,B} \ge f_0 + f_{i,A} n \ge |F_{i,n}^A|.$$

Let $B \in 2^L$. If $c_{i,B} = 0$, then $w_{AB} = 0$ and $W_{A,B} = \emptyset$ for all $A \in 2^{V(\mathbf{f}_i)}$. Otherwise,

$$nc_{i,B} = n \sum_{\substack{B_1,\ldots,B_r \\ B_i = B}} c_{B_1,\ldots,B_i} \le -m_1 + \sum_{\substack{B_1,\ldots,B_r \\ B_i = B}} |C_{B_1,\ldots,B_i}| = |G_i^B| - m_1,$$

and we have

$$\sum_{\substack{A \in 2^{V(F_{i,n})} \\ h(A) \subset B}} |W_{A,B}| \le \sum_{\substack{A \in 2^{V(F_{i,n})} \\ h(A) \subset B}} (w_{A,B} n + f_0 + 1) \le c_{i,B} n + 2^{v_0}(f_0 + 1) \le |G_i^B|.$$

Hence, we can extend $h : V_1(F_{i,n}) \to L \subset V(G_i)$ to the whole of $V(F_{i,n})$ by mapping $\cup_{h(A) \subset B} W_{A,B}$ injectively into G_i^B, which proves that $F_{i,n} \subset G_i$.

Suppose that $i \in [q+1, r]$. The relation $F_i \subset \mathbf{c}_i$ means that there exist appropriate $V_1(F_i) \cup V_2(F_i) = V(F_i)$ and $h : V_1(F_i) \to L$. We view h as a partial embedding of F_i into G_i and extend h to the whole of $V(F_i)$.

Take consecutively $y \in V_2(F_i)$. There is $B_i \subset L$ such that $c_{i,B_i} > 0$ and $h(\Gamma(y)) \subset B_i$. The inequality $c_{i,B_i} > 0$ implies that there are disjoint B_j's, $j \in [r] \setminus \{i\}$, such that $c_{B_1,\ldots,B_r} > 0$. Each vertex in $C_{B_1,\ldots,B_r}$ is connected by colour i to the whole of $B_i \supset h(\Gamma(y))$. The inequality $c_{B_1,\ldots,B_r} > 0$ means that $|C_{B_1,\ldots,B_r}| \ge m_1 \ge v(F_i)$, so we can always extend h to y, that is, we find an F_i-subgraph of colour i.

Thus the constructed graph G has the desired arrowing property, which proves the upper bound.

Let $d' = \min_{i \in [q]} \min_{x \in V(\mathbf{f}_i)} d_{\mathbf{f}_i}(x) > 0$, $l' = 5ld_0/d'$, $m_3 = \max(r^{l'}, 2^{v_0}(f_0 + l'))$. As the lower bound, we show that, for all sufficiently large n,

$$\hat{r}(F_{1,n}, \ldots, F_{q,n}, F_{q+1}, \ldots, F_r) \ge \hat{r}(\mathbf{F})n - 2^{l'} l' m_3. \tag{3.4}$$

Choose any asymptotically minimum graph G with the arrowing property. Let $L \subset V(G)$ be the set of vertices of degree at least $d'n/2$ in G. From $d'n|L|/4 < e(G) < (1 + o(1))ld_0 n$, it follows that $|L| \le l'$ (assuming that n is sufficiently large). For $A \in 2^L$, define $g_A = (|G^A| + m_3)/n$, where $G_A = \{x \in V(G) \setminus L : \Gamma(x) \cap L = A\}$.

Claim 1 $\mathbf{g} \to \mathbf{F}$.

Suppose on the contrary that there is an $\mathbf{F}$-free r-colouring $\mathbf{c}$ of $\mathbf{g}$. We are going to exhibit a contradictory r-colouring of $E(G)$.

For each $B \in 2^L$ choose any disjoint sets $C_{B_1,\ldots,B_r} \subset G^B$ (indexed by r-tuples of disjoint sets partitioning B) such that they partition G^B and

$$|C_{B_1,\ldots,B_r}| \le \lfloor c_{B_1,\ldots,B_r} \cdot n \rfloor. \tag{3.5}$$

This is possible because

$$\sum_{B_1\cup\cdots\cup B_r=B} \lfloor c_{B_1,\dots,B_r}\cdot n\rfloor \ge g_B n - r^{l'} \ge |G^B|.$$

For $j\in[r]$, $x\in B_j$ and $y\in C_{B_1,\dots,B_r}$, colour the edge $\{x,y\}\in E(G)$ by colour j. All the remaining edges of G (namely, those lying inside L or inside $V(G)\setminus L$) are coloured with colour 1.

There is $i\in[r]$ such that $G_i\subset G$, the colour-i subgraph, contains a forbidden subgraph.

Suppose that $i\in[q]$. Let $h: F_{i,n}\to G_i$ be an embedding. If n is large, then

$$d(x)\ge d'n+o(n) > d'n/2,\quad x\in V_1(F_{i,n}),$$

which implies that $h(V_1(F_{i,n}))\subset L$. Define, for $A\in 2^{V(\mathbf{f}_i)}$ and $B\in 2^L$ with $B\supset h(A)$ and $f_{i,A}\ne 0$,

$$w_{A,B}=\frac{|h^{-1}(G^B)\cap F_{i,n}^A|+f_0+l'}{n}.$$

All other $w_{A,B}$'s are set to zero. For $A\in 2^{V(\mathbf{f}_i)}$ with $f_{i,A}\ne 0$, we have

$$\sum_{\substack{B\in 2^L\\ B\supset h(A)}} w_{A,B}\ge \frac{|F_{i,n}^A\cap h^{-1}(V(G)\setminus L)|+f_0+l'}{n}\ge \frac{|F_{i,n}^A|+f_0}{n}\ge f_{i,A}.$$

For $B\in 2^L$ we have

$$\sum_{\substack{A\in 2^{V(\mathbf{f}_i)}\\ h(A)\subset B}} w_{A,B}\le \frac{2^{v_0}(f_0+l')}{n}+\sum_{\substack{A\in 2^{V(\mathbf{f}_i)}\\ h(A)\subset B}}\frac{|h^{-1}(G^B)\cap F_{i,n}^A|}{n}\le \frac{2^{v_0}(f_0+l')}{n}+\frac{|G^B|}{n}\le g_B,$$

that is, h (when restricted to $V(\mathbf{f}_i)$) and $\mathbf{w}$ demonstrate that $\mathbf{f}_i\subset \mathbf{c}_i$, which is a contradiction.

Suppose that $i\in[q+1,r]$. Let $V_1(F_i)$ consist of those vertices which are mapped by $h: F_i\to G_i$ into L and let $V_2(F_i)=V(F_i)\setminus V_1(F_i)$. This is a legitimate bipartition of F_i because any colour-i edge of G connects L to $V(G)\setminus L$. Let $y\in V_2(F_i)$. The sets $C_{B_1,\dots,B_r}$'s partition $V(G)\setminus L$; let $y\in C_{B_1,\dots,B_r}$. By (3.5) we have $c_{B_1,\dots,B_r}>0$. But $h(\Gamma(y))\subset B_i$, which shows that $F_i\subset \mathbf{g}_i$. This contradiction proves Claim 1.

Hence, $\mathbf{g}\to\mathbf{F}$ and we have

$$\hat r(\mathbf{F})\le \sum_{A\in 2^L} g_A\,|A|\le \frac{2^{l'}l'm_3}{n}+\frac{1}{n}\sum_{A\in 2^L}|G^A|\,|A|\le \frac{2^{l'}l'm_3+e(G)}{n},$$

which implies the desired inequality (3.4). ∎

A moment's thought on Claim 1 reveals the following 'characterization' of extremal graphs.

THEOREM 3.4. *Let* $\mathbf{F}$ *and the* F*'s be as in Theorem 3.3 and let*

$$G_n\to(F_{1,n},\dots,F_{q,n},F_{q+1},\dots,F_r),\quad n\in\mathbb{N},$$

be any sequence of asymptotically minimum graphs. Then there is an extremal weight $\mathbf{g}\to\mathbf{F}$ *and an increasing sequence* $(n_i)_{i\in\mathbb{N}}$ *such that, up to removing* $o(n_i)$ *edges and relabelling vertices,* G_{n_i} *can be made into a bipartite graph with* $V_1(G_{n_i})=V(\mathbf{g})$ *and* $\lim_{i\to\infty}|G_{n_i}^A|/n_i=g_A$ *for each* $A\in 2^{V(\mathbf{G})}$.

In particular, if $\mathbf{g}\to\mathbf{F}$ *is the unique extremal weight, then we can take* $n_i=i$. ∎

4. Complete Bipartite Graphs. Here we will compute asymptotically the size Ramsey number if each forbidden graph is a complete bipartite graph.

THEOREM 4.1. *Let $r \ge 2$ and $q \ge 1$. Suppose that we are given $t_1, \ldots, t_q \in \mathbb{R}_{>0}$ and $s_1, \ldots, s_r, t_{q+1}, \ldots, t_r \in \mathbb{N}$ with $t_i \ge s_i$ for $i \in [q+1, r]$. Then there exist $s \in \mathbb{N}$ and $t \in \mathbb{R}_{>0}$ such that $\mathbf{k}_{s,t} \to \mathbf{F}$ and $\hat{r}(\mathbf{F}) = e(\mathbf{k}_{s,t}) = st$, where*

$$\mathbf{F} = (\mathbf{k}_{s_1,t_1}, \ldots, \mathbf{k}_{s_q,t_q}, K_{s_{q+1},t_{q+1}}, \ldots, K_{s_r,t_r}).$$

Proof. Let us first describe an algorithm finding extremal s and t. Some by-product information gathered by our algorithm will be used in the proof of the extremality of $\mathbf{k}_{s,t} \to \mathbf{F}$.

Choose $l \in \mathbb{N}$ bigger than $\hat{r}(\mathbf{F})/t_0$, where $t_0 = \sum_{i=1}^{q} t_i$, which is the same definition as that before Lemma 3.1.

We claim that $l > \sigma$, where $\sigma = \sum_{i=1}^{r}(s_i - 1)$. Indeed, take any extremal $\mathbf{g} \to \mathbf{F}$ without isolated vertices. Lemma 3.1 implies that $d(x) \ge t_0$ for any $x \in V(\mathbf{g})$. Also, it is easy to see that $v(\mathbf{g}) > \sigma$. Hence, $l \ge \hat{r}(\mathbf{F})/t_0 \ge v(\mathbf{g}) > \sigma$, as claimed.

For each integer $s \in [\sigma+1, l]$ let $t'_s > 0$ be the infimum of $t \in \mathbb{R}$ such that $\mathbf{k}_{s,t} \to \mathbf{F}$. Also, let Π_s be the set of all sequences $\mathbf{a} = (a_1, \ldots, a_r)$ of non-negative integers with $a_i = s_i - 1$ for $i \in [q+1, r]$ and $\sum_{i=1}^{r} a_i = s$. For a sequence $\mathbf{a} = (a_1, \ldots, a_r)$ and a set A of size $\sum_{i=1}^{r} a_i$, let $\binom{A}{\mathbf{a}}$ consist of all sequences $\mathbf{A} = (A_1, \ldots, A_r)$ of sets partitioning A with $|A_i| = a_i$, $i \in [r]$.

We claim that t'_s is $sol(L_s)$, the extremal value of the following linear program L_s: "*Find $sol(L_s) = \max \sum_{\mathbf{a}\in\Pi_s} u_{\mathbf{a}}$ over all sequences $(u_{\mathbf{a}})_{\mathbf{a}\in\Pi_s}$ of non-negative reals such that*

$$\sum_{\mathbf{a}\in\Pi_s} u_{\mathbf{a}} \binom{a_i}{s_i} \le t_i \binom{s}{s_i}, \quad \text{for all } i \in [q]. \text{"} \tag{4.1}$$

Claim 1 The weight $\mathbf{k}_{s,t}$ does not arrow $\mathbf{F}$ for $t < sol(L_s)$.
To prove this, let

$$\lambda = \frac{t + sol(L_s)}{2 sol(L_s)} < 1 \quad \text{and} \quad \epsilon = \frac{1-\lambda}{2^{s+1}} \min\{t_i : i \in [q]\} > 0.$$

Let $V(\mathbf{k}_{s,t}) = [s]$. Define an r-colouring $\mathbf{c}$ of $\mathbf{k}_{s,t}$ by

$$c_{\mathbf{A}} = \frac{\lambda u_{|A_1|,\ldots,|A_r|}}{\binom{s}{|A_1|,\ldots,|A_r|}}, \quad \mathbf{a} \in \Pi_s,\ \mathbf{A} \in \binom{[s]}{\mathbf{a}}, \tag{4.2}$$

$c_{B,\emptyset,\ldots,\emptyset} = \epsilon$, $B \subsetneq [s]$, while all other c's are zero. It is indeed a colouring of $\mathbf{k}_{s,t}$:

$$\sum_{\mathbf{a}\in\Pi_s} \sum_{\mathbf{A}\in\binom{[s]}{\mathbf{a}}} c_{\mathbf{A}} = \sum_{\mathbf{a}\in\Pi_s} \lambda u_{\mathbf{a}} = \lambda\, sol(L_s) > t.$$

We have $\mathbf{k}_{s_i,t_i} \not\subset \mathbf{c}_i$ for $i \in [q]$: for example, for $i = 1$ and any $S \in \binom{[s]}{s_1}$, we have

$$\sum_{\mathbf{a}\in\Pi_s} \sum_{\substack{\mathbf{A}\in\binom{[s]}{\mathbf{a}} \\ A_1 \supset S}} c_{\mathbf{A}} = \sum_{\substack{\mathbf{a}\in\Pi_s \\ a_1 \ge s_1}} \frac{\binom{s-s_1}{a_1-s_1,a_2,\ldots,a_r}\lambda u_{\mathbf{a}}}{\binom{s}{a_1,\ldots,a_r}} = \lambda \sum_{\mathbf{a}\in\Pi_s} \frac{\binom{a_1}{s_1} u_{\mathbf{a}}}{\binom{s}{s_1}} \le \lambda t_1 < t_1 - \sum_{\substack{B\subsetneq[s] \\ B\supset S}} c_{B,\emptyset,\ldots,\emptyset}.$$

Also, $K_{s_i,t_i} \not\subset \mathbf{c}_i$ for $i \in [q+1,r]$ because $c_{A_1,\dots,A_r} = 0$ whenever $|A_i| \ge s_i$ for some $i \in [q+1,r]$. Claim 1 is proved.

Claim 2 $\mathbf{k}_{s,t} \to \mathbf{F}$ for any $t > sol(L_s)$.

Suppose that the claim is not true and we can find an $\mathbf{F}$-free r-colouring $\mathbf{c}$ of $\mathbf{k}_{s,t}$. By the definition, $c_{A_1,\dots,A_r} = 0$ whenever $|A_i| \ge s_i$ for some $i \in [q+1,r]$. If some $c_{A_1,\dots,A_r} = c > 0$ with $|A_i| \le s_i - 2$ for some $i \in [q+1,r]$, then $A_j \ne \emptyset$ for some $j \in [q]$, so we can pick $x \in A_j$ and set $c_{A_1,\dots,A_r} = 0$ while increasing $c_{\dots,A_j\setminus\{x\},\dots,A_i\cup\{x\},\dots}$ by c. Clearly, $\mathbf{c}$ remains $\mathbf{F}$-free. Thus, we can assume that all the c's are zero except those of the form $c_{\mathbf{A}}$, $\mathbf{A} \in \binom{[s]}{\mathbf{a}}$ for some $\mathbf{a} \in \Pi_s$. Now, retracing back our proof of Claim 1, we obtain a feasible solution $u_{\mathbf{a}} = \sum_{\mathbf{A}\in\binom{[s]}{\mathbf{a}}} c_{\mathbf{A}}$, $\mathbf{a} \in \Pi_s$, to L_s with a larger objective function, which is a contradiction. The claim is proved.

Thus, $t'_s = sol(L_s)$ and $m_u = \min\{st'_s : s \in [\sigma+1,l]\}$ is an upper bound on $\hat{r}(\mathbf{F})$. Let us show that in fact $\hat{r}(\mathbf{F}) = m_u$.

We rewrite the definition of $\hat{r}(\mathbf{F})$ so that we can apply the Farkas Lemma. The verification of the following easy claim is left to the reader.

Claim 3 $\hat{r}(\mathbf{F}) = \inf e(\mathbf{g})$ over all weights $\mathbf{g}$ on $L = [l]$ such that there do not exist non-negative reals $(c_{\mathbf{A}})_{\mathbf{A}\in\binom{A}{\mathbf{a}},\,\mathbf{a}\in\Pi_{|A|},\,A\in 2^L}$ with the following properties

$$\sum_{\mathbf{a}\in\Pi_{|A|}} \sum_{\mathbf{A}\in\binom{A}{\mathbf{a}}} c_{\mathbf{A}} \ge g_A, \quad A \in 2^L,$$

$$\sum_{A\in 2^L} \sum_{\mathbf{a}\in\Pi_{|A|}} \sum_{\substack{\mathbf{A}\in\binom{A}{\mathbf{a}}\\ A_i\supset S}} c_{\mathbf{A}} \le t_i, \quad i \in [q],\ S \in \binom{L}{s_i}.$$

Let $\mathbf{g}$ be any feasible solution to the above problem. By the Farkas Lemma there exist $x_A \ge 0$, $A \in 2^L$, and $y_{i,S} \ge 0$, $i \in [q]$, $S \in \binom{L}{s_i}$, such that

$$\sum_{i=1}^{q} \sum_{S\in\binom{A_i}{s_i}} y_{i,S} \ge x_A, \quad A \in 2^L,\ \mathbf{a} \in \Pi_{|A|},\ \mathbf{A} \in \binom{A}{\mathbf{a}}, \tag{4.3}$$

$$\sum_{i=1}^{q} t_i \sum_{S\in\binom{L}{s_i}} y_{i,S} < \sum_{A\in 2^L} g_A x_A. \tag{4.4}$$

We deduce that $x_A \le 0$ (and hence $x_A = 0$) if $|A| \le \sigma$ by considering (4.3) for some $\mathbf{A}$ with $|A_i| \le s_i - 1$, each $i \in [r]$.

For each A with $a := |A| > \sigma$ repeat the following. Let $(u_{\mathbf{a}})_{\mathbf{a}\in\Pi_a}$ be an extremal solution to L_a. For each $\mathbf{a} \in \Pi_a$, take the average of (4.3) over all $\mathbf{A} \in \binom{A}{\mathbf{a}}$, multiply it by $u_{\mathbf{a}}$, and add all these inequalities together to obtain the following.

$$\begin{aligned}
x_A t'_a = \sum_{\mathbf{a}\in\Pi_a} u_{\mathbf{a}} x_A \ &\le \sum_{\mathbf{a}\in\Pi_a} \frac{u_{\mathbf{a}}}{\binom{a}{a_1,\dots,a_r}} \sum_{\mathbf{A}\in\binom{A}{\mathbf{a}}} \sum_{i=1}^{q} \sum_{S\in\binom{A_i}{s_i}} y_{i,S}\\
&= \sum_{i=1}^{q} \sum_{S\in\binom{A}{s_i}} y_{i,S} \sum_{\substack{\mathbf{a}\in\Pi_a\\ a_i\ge s_i}} \frac{u_{\mathbf{a}}\binom{a-s_i}{a_1,\dots,a_{i-1},a_i-s_i,a_{i+1},\dots,a_q}}{\binom{a}{a_1,\dots,a_r}}\\
&= \sum_{i=1}^{q} \sum_{S\in\binom{A}{s_i}} y_{i,S} \sum_{\substack{\mathbf{a}\in\Pi_a\\ a_i\ge s_i}} \frac{u_{\mathbf{a}}\binom{a_i}{s_i}}{\binom{a}{s_i}} \ \le\ \sum_{i=1}^{q} t_i \sum_{S\in\binom{A}{s_i}} y_{i,S}.
\end{aligned}$$

(In the last inequality we used (4.1).)

Substituting the obtained inequalities on the x_A's into (4.4) we obtain

$$\sum_{i=1}^{q} t_i \sum_{S\in\binom{L}{s_i}} y_{i,S} < \sum_{\substack{A\in 2^L\\ |A|>\sigma}} \frac{g_A}{t'_{|A|}} \sum_{i=1}^{q} t_i \sum_{S\in\binom{A}{s_i}} y_{i,S}.$$

As the $y_{i,S}$'s are non-negative, some of these variables has a larger coefficient on the right-hand side. Let it be $y_{i,S}$. We have

$$t_i < t_i \sum_{\substack{A\in\binom{L}{>\sigma}\\ A\supset S}} \frac{g_A}{t'_{|A|}} \le \frac{t_i}{m_u} \sum_{A\in 2^L} g_A|A|. \tag{4.5}$$

The last inequality follows, by comparing coefficients at each g_A, from the fact that for any integer $a>\sigma$, we have $1/t'_a \le a/m_u$ by the definition of m_u. Hence, $e(\mathbf{g}) = \sum_{A\in 2^L} a_A|A| > m_u$ as required. ◻

COROLLARY 4.2. *Let $r\ge q\ge 1$, $t_1,\dots,t_q\in\mathbb{R}_{>0}$ and $s_1,\dots,s_r,t_{q+1},\dots,t_r\in\mathbb{N}$ with $t_i\ge s_i$ for $i\in[q+1,r]$. For $i\in[q]$, let $(t_{i,n})_{n\in\mathbb{N}}$ be an integer sequence with $t_{i,n}=t_i n+o(n)$. Define*

$$\mathbf{F}_n = (K_{s_1,t_{1,n}},\dots,K_{s_q,t_{q,n}},K_{s_{q+1},t_{q+1}},\dots,K_{s_r,t_r}).$$

Let $l\in\mathbb{N}$ be larger than $\lim_{n\to\infty}\hat{r}(\mathbf{F}_n)/(t_0 n)$, where $t_0=\sum_{i=1}^{q} t_i$. Then

$$\lim_{n\to\infty}\frac{\hat{r}(\mathbf{F}_n)}{n} = \lim_{n\to\infty}\frac{\min\{e(K_{s,t}) : s\le l,\ K_{s,t}\to\mathbf{F}_n\}}{n}. \qquad ◻ \tag{4.6}$$

In other words, in order to compute the limit in Corollary 4.2, it is sufficient to consider only complete bipartite graphs arrowing $\mathbf{F}_n$. It seems that there is no simple general formula, but the proof of Theorem 4.1 gives an algorithm for computing $\hat{r}(\mathbf{F})$. The author has realized the algorithm as a C program calling the `lp_solve 3.2` library. (The latter is a freely available linear programming software, currently maintained by Michel Berkelaar [4]). Later, David Avis rewrote the program to be linked with his `lrslib 4.1` library [1]. The latter library has the advantage that its arithmetic is exact (whilst `lp_solve` operates with reals), so that any computed limit can be considered as proved. The reader is welcome to experiment with the program; its source can be found in [15]. Here, in Table 4.1, we present the asymptotics of $\hat{r}(K_{s,n},K_{t,n})$ for $1\le s\le t\le 8$. Unfortunately, the number of iterations (which is approximately $\frac{1}{2}\lim\hat{r}(K_{s,n},K_{t,n})/n$) increases rapidly with s and t.

For certain series of parameters we can get a more explicit expression. First, let us treat the case $q=1$, that is, when only the first forbidden graph dilates with n. We can assume that $t_1=1$ by scaling n.

THEOREM 4.3. *Let $q=1$ and $r\ge 2$. Then for any $s_1,\dots,s_r,t_2,\dots,t_r\in\mathbb{N}$ with $t_i\ge s_i$, $i\in[2,r]$, we have*

$$\hat{r}(K_{s_1,n},K_{s_2,t_2},\dots,K_{s_r,t_r}) = n\cdot\min\left\{ s\,\frac{(s)_{s_1}}{(s-s')_{s_1}} : s\in\mathbb{N}_{>\sigma}\right\} + O(1),$$

where $s'=\sigma-s_1+1$, $\sigma=\sum_{i=1}^{r}(s_i-1)$ and $(s)_k = s(s-1)\dots(s-k+1)$.

Proof. The Problem L_s has only one variable $u_{s-s',s_2-1,\dots,s_r-1}$. Trivially, $t'_s = \binom{s}{s_1}/\binom{s-s'}{s_1} = (s)_{s_1}/(s-s')_{s_1}$, and the theorem follows. ◻

TABLE 4.1
Values of $\lim_{n\to\infty} \hat{r}(K_{s,n}, K_{t,s})/n$ *obtained with the* `lrslib` *library of David Avis.*

1	2							
2	6	18						
3	12	40	98					
4	20	75	$182\frac{14}{19}$	363				
5	30	$118\frac{10}{17}$	$310\frac{19}{62}$	$638\frac{44}{47}$	1156			
6	42	$172\frac{4}{5}$	$469\frac{6}{7}$	$1023\frac{23}{87}$	$1952\frac{15}{22}$	$3350\frac{1}{3}$		
7	56	$241\frac{7}{23}$	$678\frac{4}{11}$	$1538\frac{36}{55}$	$3030\frac{1}{2}$	$5456\frac{92}{209}$	$9120\frac{42}{55}$	
8	72	$320\frac{4}{7}$	$938\frac{2}{5}$	$2211\frac{579}{1573}$	$4517\frac{317}{504}$	$8426\frac{176}{221}$	$14523\frac{595}{4693}$	$23781\frac{7}{34}$
(s,t)	1	2	3	4	5	6	7	8

In the case $s_1 = 1$ we obtain the following formula.

COROLLARY 4.4. *For any* $s_2, \ldots, s_r, t_2, \ldots, t_r \in \mathbb{N}$ *with* $t_i \geq s_i$, $i \in [2, r]$, *we have*

$$\hat{r}(K_{1,n}, K_{s_2,t_2}, \ldots, K_{s_r,t_r}) = 4\left(1 - r + \sum_{i=2}^{r} s_i\right) n + O(1).$$

Proof. By Theorem 4.3, we have to compute $\min_{s>s'} \frac{s^2}{s-s'}$, where $s' = \sum_{i=2}^{r}(s_i - 1)$. The differentiation $\frac{d}{ds}(\frac{s^2}{s-s'}) = \frac{s(s-2s')}{(s-s')^2}$ shows that the minimum is attained for $s = 2s'$. ◻

Another case with a simple formula for $\hat{r}(\mathbf{F})$ is $q = 2$, $s_1 = s_2$, and $t_1 = t_2$. Again, without loss of generality we can assume that $t_1 = t_2 = 1$.

THEOREM 4.5. *Let* $q = 2$ *and* $r \geq 2$. *Then for any* $s, s_3, \ldots, s_r, t_3, \ldots, t_r \in \mathbb{N}$ *with* $t_i \geq s_i$, $i \in [3, r]$, *we have*

$$\hat{r}(K_{s,n}, K_{s,n}, K_{s_3,t_3}, \ldots, K_{s_r,t_r}) = n \cdot \min\{a \cdot f(a) : a \in \mathbb{N}_{>\sigma}\} + O(1), \tag{4.7}$$

where $\sigma = 2s - r + \sum_{i=3}^{r} s_i$ *and*

$$f(a) = \frac{2\binom{a}{s}}{\binom{\lfloor a'/2\rfloor}{s} + \binom{\lceil a'/2\rceil}{s}},$$

with $a' = a - \sum_{i=3}^{r}(s_i - 1)$.

Proof. Let $a \in \mathbb{N}_{>\sigma}$ and let $(u_{\mathbf{a}})_{\mathbf{a}\in\Pi_a}$ be an extremal solution to L_a. (Where we obviously define $s_1 = s_2 = s$ and $t_1 = t_2 = 1$.) Excluding the constant indices in $u_{\mathbf{a}}$, we assume that the index set Π_a consists of pairs of integers (a_1, a_2) with $a_1 + a_2 = a'$.

Clearly, $(u'_{a_1,a_2})_{(a_1,a_2)\in\Pi_a}$ is also an extremal solution, where $u'_{a_1,a_2} = \frac{1}{2}(u_{a_1,a_2} + u_{a_2,a_1})$, Thus we can assume that $u_{a_1,a_2} = u_{a_2,a_1}$ for all $(a_1, a_2) \in \Pi_a$.

If $u_{a_1,a_2} = c > 0$ for some $a_1 < \lfloor a'/2\rfloor$, then we can set $u_{a_1,a_2} = u_{a_2,a_1} = 0$, while increasing $u_{\lfloor a'/2\rfloor,\lceil a'/2\rceil}$ and $u_{\lceil a'/2\rceil,\lfloor a'/2\rfloor}$ by c. The easy inequality

$$\binom{b+1}{s} + \binom{a'-b-1}{s} - \binom{b}{s} - \binom{a'-b}{s} = \binom{b}{s-1} - \binom{a'-b-1}{s-1} < 0, \quad s - 1 \leq b < \lfloor a'/2\rfloor$$

implies inductively that the left-hand side of (4.1) strictly decreases while the objective function $\sum_{\mathbf{a}\in\Pi_a} u_{\mathbf{a}}$ does not change, which clearly contradicts the minimality of $\mathbf{u}$.

Now we deduce that, for *any* extremal solution $(u_{\mathbf{a}})_{\mathbf{a}\in\Pi_a}$, we have $u_{a_1,a_2}=0$ unless $\{a_1,a_2\}=\{\lfloor a'/2\rfloor,\lceil a'/2\rceil\}$; moreover, it follows that necessarily $u_{\lfloor a'/2\rfloor,\lceil a'/2\rceil}=u_{\lceil a'/2\rceil,\lfloor a'/2\rfloor}$. Hence, $t'_a=f(a)$, which proves the theorem. □

The special case $r=2$ of Theorem 4.5 answers the question of Erdős, Faudree, Rousseau and Schelp [8, Problem B], who asked for the value of

$$r_s=\lim_{n\to\infty}\frac{\hat r(K_{s,n},K_{s,n})}{n}.$$

The formula (4.7), which now reads $r_s=\min_{a\ge 2s-1} af(a)$ with $f(a)=2\binom{a}{s}/(\binom{\lfloor a/2\rfloor}{s}+\binom{\lceil a/2\rceil}{s})$, can be further simplified in this case as follows.

Theorem 4.6. *For $s\ge 4$ we have $r_s=a_sf(a_s)$, where $a_s=2\lfloor s(s+3)/4\rfloor-3$.*

Proof. For any $b\ge s$ we have $f(2b)=f(2b-1)$; hence, the minimum of $af(a)$ is attained for an odd a:

$$r_s=\min_{b\ge s}\,(2b-1)f(2b-1)=2\min_{b\ge s}\,(2b-1)\binom{2b-1}{s}\left(\binom{b-1}{s}+\binom{b}{s}\right)^{-1}.$$

We have $\binom{b-1}{s}+\binom{b}{s}=\frac{(b-1)!(2b-s)}{s!(b-s)!}$ and, as it is routine to check,

$$(2b+1)f(2b+1)-(2b-1)f(2b-1)=cp_s(b),$$

where $c=\frac{2(2b-1)!(b-s)!}{(2b-s+1)!(b-1)!(2b-s+2)}$ and

$$p_s(b)=2(2b+1)^2(b-s+1)-(2b-1)(2b-s+1)(2b-s+2)=8b^2-2bs^2-6bs+12b+s^2-5s+4.$$

The quadratic in b polymomial p_s has two roots: one is less than 1 (because $p_s(1)<0$) and the other is bigger than s (because $p_s(s)<0$). Thus, the function $(2b-1)f(2b-1)$, $b\ge s$, first decreases and then increases; its minimum is attained for b_s, the smallest integer $b\ge s$ with $p_s(b)>0$. The value of b_s can be computed exactly:

$$b_s=\begin{cases}4t^2+3t-1, & s=4t,\\ 4t^2+5t, & s=4t+1,\\ 4t^2+7t+1, & s=4t+2,\\ 4t^2+9t+3, & s=4t+3.\end{cases}$$

For example, let us check the case $s\equiv 0 \pmod 4$:

$$p_s(4t^2+3t-2)=-32t+12<0<p_s(4t^2+3t-1)=32t^2-8t.$$

Also, $2b_s-1=2\lfloor s(s+3)/4\rfloor-3$ in each case, as required. □

Remark. For $4\le s\le 8$, the values of r_s given by Table 4.1 and Theorem 4.6 coincide, which is reassuring.

The natural question, to characterise all extremal weights in Theorem 4.1, arises. We have a partial answer as follows. Let $\mathbf{g}\to\mathbf{F}$ be extremal. We know by Lemma 3.1 that $v(\mathbf{g})\le l$, so assume that $v(\mathbf{g})\subset[l]$. It is easy to check that if we increase each g_A by some $\epsilon>0$ then the obtained weight is a feasible solution to the system of Claim 3 from the proof of Theorem 4.1 and hence satisfies (4.5) for some i and S. As

$\epsilon > 0$ is arbitrary and there finitely many possible pairs (i, S), the weight $\mathbf{g}$ satisfies the *non-strict* inequality (4.5) for some (i, S). As $\mathbf{g}$ is extremal, we have, in fact, an equality there. This implies that $g_A = 0$ unless $|A| \cdot t'_{|A|} = m_u$ and $A \supset S$.

But in some cases we can get more precise information. As an example, consider $\mathbf{F} = (\mathbf{k}_{2,1}, \mathbf{k}_{2,1})$. Theorem 4.5 implies that $\hat{r}(\mathbf{F}) = 18$. But we are able to show more.

THEOREM 4.7. $\mathbf{k}_{3,6} \to (\mathbf{k}_{2,1}, \mathbf{k}_{2,1})$ *is the unique extremal weight. Also, there is* n_0 *such that, for all* $n > n_0$*, we have* $\hat{r}(K_{2,n}, K_{2,n}) = 18n - 15$ *and* $K_{3,6n-5}$ *and* $K_3 + \overline{K}_{6n-6}$ *are the only extremal graphs (up to isolated vertices).*

Proof. Let $\mathbf{g} \to (\mathbf{k}_{2,1}, \mathbf{k}_{2,1})$ have size 18 and no isolated vertices.

By Lemma 3.1 we have $v(\mathbf{g}) \le 9$. It is routine to check that $at'_a > 18$ for any $a \in [4, 9]$. Thus we know that, for some $S = \{x, y\} \subset L$, we have $g_A = 0$ whenever $|A| \ne 3$ or $A \not\supset S$. Let J be the set of those $j \in L$ with $g_{\{x,y,j\}} > 0$. We have $\sum_{j \in J} g_{\{x,y,j\}} = 6$. Suppose, on the contrary to the claim, that $\mathbf{g} \not\cong \mathbf{k}_{3,6}$. Then we have $|J| \ge 2$.

Consider the 2-colouring $\mathbf{c}$ of $\mathbf{g}$ obtained by letting $c_{A_1,A_2} = 2^{-18}/10$ for all disjoint $A_1, A_2 \in 2^L$ except

$$\left.\begin{array}{rclclcl} c_{\{x,j\},\{y\}} &=& c_{\{y,j\},\{x\}} &=& c_{\{x\},\{y,j\}} = c_{\{y\},\{x,j\}} &=& 0.9,\\ c_{\{x,y\},\{j\}} &=& c_{\{j\},\{x,y\}} &=& (g_{\{x,y,j\}} - 3.5)_+/2, && \end{array}\right. \qquad j \in J,$$

where $f_+ = f$ if $f > 0$ and $f_+ = 0$ if $f \le 0$. This is a colouring of $\mathbf{g}$: for example,

$$\sum_{A_1 \cup A_2 = \{x,y,j\}} c_{A_1,A_2} > 4 \times 0.9 + 2 \times (g_{\{x,y,j\}} - 3.5)_+/2 > g_{\{x,y,j\}}.$$

Also, neither $\mathbf{c}_1$ nor $\mathbf{c}_2$ contains $\mathbf{k}_{2,1}$: for example,

$$\sum_{\substack{A \in 2^L \\ A \supset \{x,y\}}} c_{i,A} < (5 - 3.5)/2 + 0.1 < 1, \quad i = 1, 2,$$

as $d_{\mathbf{g}}(j) \ge 1$, $j \in J$. This contradiction proves the first claim.

Let G_n be a minimum $(K_{2,n}, K_{2,n})$-arrowing graph and let $L_n = \{x \in V(G_n) : d(x) \ge n\}$. By Theorem 3.4 $|L_n| = 3$ for all large n. By the minimality of G_n, $V(G_n) \setminus L_n$ spans no edge and each $x \in V(G_n) \setminus L_n$ sends at least 3 edges to L_n. (In particular, $|L_n| = 3$.)

If L spans one or two edges in G_n, then these edges can be removed without affecting the arrowing property. Thus $e(G_n[L_n])$ equals 0 or 3. Now, the easy analysis completes the proof. ∎

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