Lattice Points in Lattice Polytopes

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Abstract

We show that, for any lattice polytope $P \subset \mathbb{R}^d$, the set $\mathrm{int}(P) \cap l\mathbb{Z}^d$ (provided it is non-empty) contains a point whose coefficient of asymmetry with respect to P is at most $8d \cdot (8l+7)^{2^{2d+1}}$. If, moreover, P is a simplex, then this bound can be improved to $8 \cdot (8l+7)^{2^{d+1}}$.

As an application, we deduce new upper bounds on the volume of a lattice polytope, given its dimension and the number of sublattice points in its interior.

1 Introduction

A *lattice polytope* in $\mathbb{R}^d$ is a convex polytope whose vertices are lattice points, that is, points in $\mathbb{Z}^d$. For an integer $l \geq 1$, let $I_l(P) = \mathrm{int}(P) \cap l\mathbb{Z}^d$ be the set of interior points of P whose coordinates are integers divisible by l.

Of course, some points of $I_l(P)$ can lie 'close' to ∂P, the boundary of P. However, our Theorem 4 shows that, provided $I_l(P) \neq \emptyset$, there is $\mathbf{w} \in I_l(P)$ with

$$\mathrm{ca}(\mathbf{w}, P) \leq 8d \cdot (8l+7)^{2^{2d+1}}, \tag{1}$$

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where $\mathrm{ca}(\mathbf{w}, P)$ is the *coefficient of asymmetry* of P about $\mathbf{w}$:

$$\mathrm{ca}(\mathbf{w}, P) = \max_{|\mathbf{y}|=1} \frac{\max\{\lambda \mid \mathbf{w} + \lambda \mathbf{y} \in P\}}{\max\{\lambda \mid \mathbf{w} - \lambda \mathbf{y} \in P\}}.$$

Although the function in the right-hand side of (1) is enormous, the main point is that it depends only on d and l.

We prove an inequality of this type for the case of a simplex S first. Namely, Theorem 2 implies that, for some $\mathbf{w} \in I_l(S)$,

$$\mathrm{ca}(\mathbf{w}, S) \leq 8 \cdot (8l + 7)^{2^{d+1}}. \tag{2}$$

Here the claim essentially concerns the barycentric coordinates $(\alpha_0, \ldots, \alpha_d)$ of $\mathbf{w}$ inside S because of the easy relation

$$\mathrm{ca}(\mathbf{w}, S) = \max_{0 \leq i \leq d} \frac{1 - \alpha_i}{\alpha_i} = \frac{1}{m_S(\mathbf{w})} - 1, \tag{3}$$

where $m_S(\mathbf{w}) := \min_{0 \leq i \leq d} \alpha_i$ is the smallest barycentric coordinate of $\mathbf{w} \in S$. Define

$$\beta(d, l) := \inf_S \max\{m_S(\mathbf{w}) \mid \mathbf{w} \in I_l(S)\}, \tag{4}$$

where the infimum is taken over all lattice simplices S with $I_l(S) \neq \emptyset$. (For example, it is easy to see that $\beta(1, l) = \frac{1}{l+2}$.) Thus we have to prove a positive lower bound on $\beta(d, l)$. The gist the proof is that if we have $\mathbf{w} \in I_l(S)$ with $m_S(\mathbf{w})$ being 'small', then using one approximation lemma of Lagarias and Ziegler [3] we can 'jump' to another vertex $\mathbf{w}' \in I_l(S)$ with $m_S(\mathbf{w}') > m_S(\mathbf{w})$, see Theorem 2.

In fact, one result of Lawrence [4, Theorem 3] implies that $\beta(d, 1) > 0$ but does not give any explicit bound, see Section 7 here.

It would be interesting to know how far our bounds (1) and (2) are from the best possible values. The best values that we know arise from the following family of lattice simplices.

Define inductively the sequence $t_{d,l}$ by $t_{1,l} = l+1$ and $t_{d+1,l} = t_{d,l}^2 - t_{d,l} + 1$. (This sequence appears in [3].) Consider the simplex

$$B_{d,l} := \mathrm{conv}\{t_{1,l}\,\mathbf{e}_1,\ \ldots,\ t_{d-1,l}\,\mathbf{e}_{d-1},\ t_{d,l}\,\mathbf{e}_d,\ -\mathbf{e}_d\} \subset \mathbb{R}^d, \tag{5}$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is the standard basis. It is not hard to see that $I_l(B_{d,l}) = \{l\mathbf{1}, l(\mathbf{1}-\mathbf{e}_d)\}$, cf. [3, Proposition 2.6]. We have $m_{B_{d,l}}(l\mathbf{1}) = m_{B_{d,l}}(l(\mathbf{1}-\mathbf{e}_d)) = \frac{l}{t_{d,l}^2-1}$: the vertex $l\mathbf{1} = (l, \ldots, l)$, for example, has barycentric coordinates

$$\left(\frac{l}{t_{1,l}},\ \ldots,\ \frac{l}{t_{d-1,l}},\ \frac{l}{t_{d,l}-1} \times \frac{t_{d,l}}{t_{d,l}+1},\ \frac{l}{t_{d,l}-1} \times \frac{1}{t_{d,l}+1}\right).$$

One can show that $t_{d,l} \geq (l+1)^{2^{d-2}}+1$ for $d \geq 2$ by considering $u_{d,l} = t_{d,l}-1$; hence

$$\beta(d,l) \leq \frac{l}{t_{d,l}^2 - 1} \leq l(l+1)^{-2^{d-1}}, \quad d \geq 2. \tag{6}$$

Thus (2) establishes the correct type of dependence on d and l, although the gap between the bounds is huge. Perhaps, $B_{d,l}$ gives the actual value of the function $\beta(d,l)$ as well as the sharp bound for (1).

To extend Theorem 2 to a general lattice polytope $P \subset \mathbb{R}^d$ (Theorem 4), we try to find a lattice polytope $P' \subset P$ with few vertices such that $I_l(P') \neq \emptyset$ and a homothetic copy of P' covers P. The latter condition gives an upper bound on $\mathrm{ca}(\mathbf{w}, P)$ in terms of $\mathrm{ca}(\mathbf{w}, P')$ for $\mathbf{w} \in \mathrm{int}(P')$, see Lemma 3, and is satisfied if, for example, $P' \supset S$, where $S \subset P$ is a simplex of the maximum volume. But to get a non-empty $I_l(P')$ we may have to add as many as d extra vertices to S. It is now possible to define our jumps within P' to get the required $\mathbf{w} \in I_l(P')$. However, the bound (1) for d-polytopes that we obtain is comparable with that for $2d$-dimensional simplices; we believe that we lose here too much but we have not found any better argument.

Next, we investigate the following problem. Let $p(d,k,l)$ (resp. $s(d,k,l)$) be the maximum volume of a lattice polytope (resp. simplex) $P \subset \mathbb{R}^d$ with $|I_l(P)| = k$. As for any $d \geq 2$ there exist lattice simplices with no lattice points in the interior and of arbitrarily large volume, we restrict our consideration to the case $k \geq 1$.

Trivially, $p(1,k,l) = s(1,k,l) = (k+1)l$. A result of Scott [7] implies that $p(2,1,1) = s(2,1,1) = \frac{9}{2}$ and $p(2,k,1) = s(2,k,1) = 2(k+1)$ for $k \geq 2$. Hensley [2, Theorem 3.6] showed that $p(d,k,1)$ exists (i.e., it is finite) for $k \geq 1$. The method of Hensley was sharpened by Lagarias and Ziegler [3, Theorem 1], who showed that

$$p(d,k,l) \leq kl^d(7(kl+1))^{d2^{d+1}}, \tag{7}$$

and also observed that, for any fixed (d,k,l), there are finitely many (up to a $GL_n(\mathbb{Z})$-equivalence) lattice polytopes $P \subset \mathbb{R}^d$ with $|I_l(P)| = k \geq 1$.

Lagarias and Ziegler [3, Theorem 2.5] proved the following extension of a theorem of Mahler [5]: "A convex body $K \subset \mathbb{R}^d$ with $k = |I_l(K)| \geq 1$ satisfies

$$\mathrm{vol}(K) \leq (l(\mathrm{ca}(\mathbf{w},K)+1))^d \cdot k, \tag{8}$$

for any $\mathbf{w} \in I_l(K)$."

Combining (8) with (1) (or more exactly with (27)), we obtain that

$$p(d,k,l) \leq (8dl)^d \cdot (8l+7)^{d \cdot 2^{2d+1}} \cdot k. \tag{9}$$

A theorem of Blichfeldt [1] says that $|P \cap \mathbb{Z}^d| \leq d + d!\,\mathrm{vol}(P)$; combined with (9) it gives an upper bound on $|P \cap \mathbb{Z}^d|$ in terms of $|I_l(P)|$ (if the latter set is non-empty).

An upper bound on $s(d,k,l)$ can be obtained by applying (8) to (2). However, we obtain a better bound in Theorem 6 by exploiting the geometry of a simplex, namely we show that

$$s(d,k,l) \leq 2^{3d-2} \cdot l^d \cdot (8l+7)^{(d-1)2^{d+1}} \cdot k/d!. \tag{10}$$

The best lower bound on $p(d,k,l)$ and $s(d,k,l)$ that we know (except for $(d,k,l) = (2,1,1)$), comes from the consideration of the simplex

$$S_{d,k,l} := \mathrm{conv}\{\mathbf{0},\, t_{1,l}\,\mathbf{e}_1,\ \ldots,\, t_{d-1,l}\ \mathbf{e}_{d-1}, (k+1)(t_{d,l}-1)\mathbf{e}_d\},$$

which satisfies $I_l(S_{d,k,l}) = \{l\mathbf{1} + il\mathbf{e}_d \mid 0 \leq i \leq k-1\}$, see [3, Proposition 5.6]. This demonstrates that

$$s(d,k,l) \geq \mathrm{vol}(S_{d,k,l}) > \frac{k+1}{d! \cdot l}\,(l+1)^{2^{d-1}},$$

see formula (2.13) in [3]. The family $(S_{d,k,1})$ was found by Zaks, Perles and Wills [9] and its generalization (the addition of parameter l)—by Lagarias and Ziegler [3].

Again, we have the correct type of dependence of d, k and l but the gap between the known bounds is huge. The ultimate aim would be to find exact values, which is probably not hopeless because the above contructions, believed to be extremal, are rather simple.

2 Jumping inside a simplex

We will use the following lemma of Lagarias and Ziegler [3, Lemma 2.1].

Lemma 1 *For a real $\lambda \geq 1$ and integer $n \geq 1$, define*

$$\delta(n, \lambda) = (7(\lambda + 1))^{-2^{n+1}}. \tag{11}$$

Then, for all positive real numbers $\alpha_1, \dots, \alpha_n$ satisfying

$$1 - \delta(n, \lambda) < \sum_{i=1}^{n} \alpha_i \leq 1,$$

there exist non-negative integers $P_1, \dots, P_n, Q$ such that

$$Q = P_1 + \cdots + P_n > 0, \tag{12}$$

$$\alpha_i > \frac{\lambda P_i}{\lambda Q + 1} \quad \text{for } 1 \leq i \leq n, \tag{13}$$

$$\lambda Q + 1 \leq \delta(n, \lambda)^{-1}. \blacksquare \tag{14}$$

The above lemma is the main ingredient in our 'jumps.' Here it is applied with $\lambda = \frac{8}{7} l$. There is nothing special about the constant $\frac{8}{7}$ except it makes (11) look simpler; any fixed number greater than 1 would do as well.

Theorem 2 *Let $l \geq 1$ and let $S = \mathrm{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_d\} \subset \mathbb{R}^n$ be a lattice simplex. If* rel-int$(S) \cap l\mathbb{Z}^n$ *is non-empty, then it contains a point $\mathbf{w}$ with*

$$m_S(\mathbf{w}) \geq \gamma := \delta(d, \tfrac{8}{7} l)/8 = (8l + 7)^{-2^{d+1}}/8. \tag{15}$$

Proof. We may assume that $n = d$ because we can always find a linear transformation preserving the lattice $\mathbb{Z}^n$ (and so $l\mathbb{Z}^n$ as well) and mapping S into $\mathbb{R}^d \subset \mathbb{R}^n$.

Let $\mathbf{w} = \sum_{i=0}^{d} \alpha_i \mathbf{v}_i \in I_l(S)$, $\sum_{i=0}^{d} \alpha_i = 1$, be a vertex maximizing $m_S(\mathbf{w})$. Suppose that the claim is not true. Assume that $\alpha_0 \leq \cdots \leq \alpha_d$; then $m_S(\mathbf{w}) = \alpha_0 < \gamma$. Let j be the index with $\alpha_j < 8\gamma \leq \alpha_{j+1}$; note that $j \leq d-1$ is well-defined as $\alpha_d \geq \frac{1}{d+1} \geq 8\gamma$.

We have $\sum_{i=0}^{j} \alpha_j < 8\gamma(j+1)$ which, as it is easy to see, does not exceed $\delta(d-j, \frac{8}{7} l)$ for $j \in [0, d-1]$. Hence, Lemma 1 is applicable to the $d-j$ numbers $\alpha_{j+1}, \dots, \alpha_d$ and yields integers $P_{j+1}, \dots, P_d, Q$ satisfying (12)–(14).

Consider the vertex

$$\mathbf{w}' = (lQ+1)\mathbf{w} - \sum_{i=j+1}^{d} lP_i\mathbf{v}_i \ \in \ l\mathbb{Z}^d.$$

We have $\mathbf{w}' = \sum_{i=0}^{d} \alpha_i'\mathbf{v}_i$, where, for $i \in [0,j]$, $\alpha_i' := (lQ+1)\alpha_i > \alpha_0$ and, for $i \in [j+1,d]$, $\alpha_i' := (lQ+1)\alpha_i - lP_i > \alpha_i/8 \geq \alpha_0$ by (13). As

$$\sum_{j=0}^{n} \alpha_j' = (lQ+1)\sum_{i=0}^{d}\alpha_i - \sum_{i=j+1}^{d} lP_i = 1,$$

the lattice point $\mathbf{w}'$ lies in the interior of S and contradicts the choice of $\mathbf{w}$. ∎

Remark. For $n = d$ the inequality (2) claimed in the introduction follows by applying (3) to the vertex $\mathbf{w} \in I_l(S)$ given by Theorem 2

3 $\beta(d,l)$ for small d and l

Let us try to deduce some estimates of $\beta(d,l)$ when d and l are small. We have a general upper bound (6) which in particular says that

$$\beta(2,l) \ \leq \ \frac{l}{t_{2,l}^2 - 1} \ = \ \frac{1}{(l+1)(l^2+l+2)}, \tag{16}$$

$$\beta(3,1) \ \leq \ \frac{1}{48}, \quad \beta(3,2) \ \leq \ \frac{1}{924}. \tag{17}$$

Here we present some results obtained with the help of computer showing that (16) and (17) are probably sharp.

How can one get a lower bound on, for example, $\beta(2,1)$? Our approach is the following.

Given a lattice simplex $S = \mathrm{conv}\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^2$, let $\mathbf{w}$ be a lattice vertex maximizing m_S over $I_1(S) \neq \emptyset$. Write the barycentric representation $\mathbf{w} = \sum_{i=0}^{2} \alpha_i\mathbf{v}_i$. We have

$$\alpha_0 + \alpha_1 + \alpha_2 = 1. \tag{18}$$

Without loss of generality we may assume that

$$0 \leq \alpha_0 \leq \alpha_1 \leq \alpha_2. \tag{19}$$

Consider the vertex $\mathbf{w}' = 2\mathbf{w} - \mathbf{v}_2$ which is the jump of $\mathbf{w}$ corresponding to $\mathbf{P} = (0, 0, 1)$. Its barycentric coordinates $(\alpha'_0, \alpha'_1, \alpha'_2) = (2\alpha_0, 2\alpha_1, 2\alpha_2 - 1)$ satisfy $\alpha'_1 \geq \alpha'_0 > \alpha_0$. By the choice of $\mathbf{w}$, we must have

$$2\alpha_2 - 1 \leq \alpha_0. \tag{20}$$

Similarly, the $(0, 1, 1)$-jump $\mathbf{w}' = 3\mathbf{w} - \mathbf{v}_1 - \mathbf{v}_2$ satisfies $\alpha'_0 = 3\alpha_0 > \alpha_0$ and, in the view of $\alpha'_2 \geq \alpha'_1$, we obtain

$$3\alpha_1 - 1 \leq \alpha_0. \tag{21}$$

Not everything goes so smoothly if we consider e.g. the $(0, 1, 2)$-jump, when we can only deduce that

$$4\alpha_1 - 1 \leq \alpha_0 \quad \text{or} \quad 4\alpha_2 - 2 \leq \alpha_0. \tag{22}$$

How small can α_0 be, given only the constraints (18)–(22) (which can be realized as a mixed integer program)? Solving this MIP, we obtain that $\alpha_0 \geq \frac{2}{19}$. Knowing this bound, we can enlarge our arsenal of jumps. For example, the vertex $\mathbf{w}' = 12\mathbf{w} - \mathbf{v}_0 - 4\mathbf{v}_1 - 6\mathbf{v}_2$ satisfies $\alpha'_0 = 12\alpha_0 - 1 > \alpha_0$; hence

$$12\alpha_1 - 4 \leq \alpha_0 \quad \text{or} \quad 12\alpha_2 - 6 \leq \alpha_0. \tag{23}$$

The addition of (23) to the system (18)–(22), improves our lower bound to $\alpha_0 \geq \frac{2}{17}$.

In this manner we can repeatedly add new constraints to our MIP as long as this improves the lower bound on α_0. This was realized as a program in C which can be linked with either CPLEX (commercial) or `lp_solve` (public domain) MIP solver. The source code is freely available [6] and the reader is welcome to experiment with it.

We ran the program, with CPLEX 6.6, for various d and l; Table 3 records the obtained results rounded to 10^{-6}. (The column "i" denotes the number of iterations before the lower bound is obtained.) If the calculations were exact, then the fifth column would give a lower bound on $\beta(d, l)$. We consider the obtained data not as a proof but rather as an empirical evidence, although some estimation of the calculation errors is done in [6].

Unfortunately, we had no success for other pairs (d, l): the obtained lower bound was still zero when the MIP became too large to solve.

d	l	upper bound	i	output
2	1	0.125000	375	0.124906
2	2	0.041667	167	0.041648
2	3	0.017858	72	0.017850
2	4	0.009091	38	0.009086
2	5	0.005209	33	0.005205
2	6	0.003247	40	0.003245
2	7	0.002156	49	0.002150
2	8	0.001502	65	0.001499
2	9	0.001087	82	0.001085
2	10	0.000812	97	0.000810
2	11	0.000622	116	0.000621
2	12	0.000487	144	0.000486
2	13	0.000389	162	0.000387
2	14	0.000315	186	0.000314
2	15	0.000259	222	0.000258
2	16	0.000215	258	0.000214
2	17	0.000181	278	0.000180
2	18	0.000153	314	0.000152
2	19	0.000131	345	0.000130
2	20	0.000113	402	0.000112
2	21	0.000098	435	0.000097
2	22	0.000086	461	0.000085
2	23	0.000076	497	0.000073
2	24	0.000067	573	0.000065
2	25	0.000059	597	0.000058
2	26	0.000053	736	0.000051
2	27	0.000048	804	0.000046
3	1	0.020834	381	0.020795
3	2	0.001083	423	0.001077

Table 1: Computed lower bounds on $\beta(d,l)$ using CPLEX.

4 Extending results to lattice polytopes

First we have to express analytically the intuitively obvious fact that if two polytopes cover each other (up to a small homothety) then their coefficients of asymmetry cannot be far apart.

Lemma 3 *Let $P' \subset P$ be two polytopes such that P can be covered by a translate of $\lambda P'$. Then, for any $\mathbf{w} \in \mathrm{int}(P')$,*

$$\mathrm{ca}(\mathbf{w}, P) \leq |\lambda| \,\mathrm{ca}(\mathbf{w}, P') + |\lambda| - 1. \tag{24}$$

Proof. Assume that $|\lambda| > 1$, for otherwise $P' = P$ and we are home. Also, the case of 1-dimensional polytopes is trivial.

Let $\mathbf{w}_1, \mathbf{w}_2 \in \partial P$ be two points with $\mathbf{w} \in [\mathbf{w}_1, \mathbf{w}_2]$ and $\mathrm{ca}(\mathbf{w}, P) = |\mathbf{w}_1 - \mathbf{w}| : |\mathbf{w} - \mathbf{w}_2|$; let $\partial P' \cap [\mathbf{w}_i, \mathbf{w}] = \{\mathbf{w}'_i\}$, $i = 1, 2$, where $[\mathbf{x}, \mathbf{y}]$ denotes the straight line segment between $\mathbf{x}$ and $\mathbf{y}$. Clearly,

$$|\mathbf{w}_1 - \mathbf{w}_2| = (\mathrm{ca}(\mathbf{w}, P) + 1)\, |\mathbf{w} - \mathbf{w}_2| \geq (\mathrm{ca}(\mathbf{w}, P) + 1)\, |\mathbf{w} - \mathbf{w}'_2|. \tag{25}$$

As $\lambda P'$ covers $\{\mathbf{w}_1, \mathbf{w}_2\}$, there are $\mathbf{u}_1, \mathbf{u}_2 \in P'$ with

$$\mathbf{w}_1 - \mathbf{w}_2 = |\lambda| \, (\mathbf{u}_1 - \mathbf{u}_2). \tag{26}$$

We can assume that $\mathbf{u}_1, \mathbf{u}_2 \in \partial P'$. If $\mathbf{u}_1 = \mathbf{w}_1'$ and $\mathbf{u}_2 = \mathbf{w}_2'$, then we let $\mathbf{v} = \mathbf{u}_2$. Otherwise let $\mathbf{v}$ be the (well-defined, possibly "infinitely remote") point of intersection of $L(\mathbf{u}_1, \mathbf{w})$ and $L(\mathbf{u}_2, \mathbf{w}_2')$, where $L(\mathbf{x}, \mathbf{y})$ denotes the line though the points $\mathbf{x}$ and $\mathbf{y}$. As $\mathbf{v} \in L(\mathbf{u}_2, \mathbf{w}_2')$ lies outside of $\operatorname{int}(P')$, we have

$$\operatorname{ca}(\mathbf{w}, P') \geq \frac{|\mathbf{u}_1 - \mathbf{w}|}{|\mathbf{w} - \mathbf{v}|} = \frac{|\mathbf{u}_1 - \mathbf{u}_2| - |\mathbf{w} - \mathbf{w}_2'|}{|\mathbf{w} - \mathbf{w}_2'|},$$

which implies the required by (25) and (26). ∎

Remark. Note that the bound in (24) is sharp, as is demonstrated e.g. by $C_d \subset |\lambda| C_d$ and $\mathbf{w} = c\mathbf{1}$ with $0 < c \leq \frac{1}{2}$, where $C_d \subset \mathbb{R}^d$ is the 0/1-cube.

Now we are ready to prove our result on lattice polytopes.

Theorem 4 *Let $l \geq 1$ be an integer and let $P \subset \mathbb{R}^d$ be a lattice polytope with $I_l(P) \neq \emptyset$. Then there is $\mathbf{w} \in I_l(P)$ with*

$$\operatorname{ca}(\mathbf{w}, P) \leq \frac{8d}{\delta(2d, \frac{8}{7}\, l)} - 1 = 8d \cdot (8l+7)^{2^{2d+1}} - 1. \tag{27}$$

Proof. Let $S = \operatorname{conv}\{\mathbf{v}_0, \dots \mathbf{v}_d\} \subset P$ be a simplex of the maximum volume; we may assume that each $\mathbf{v}_i$ is a vertex of P.

Choose $\mathbf{u} \in I_l(P)$. Let $\mathbf{u}_1 \in \operatorname{int}(S)$ be any vertex and let $\mathbf{u}_2$ be the point of intersection of the ray $\{\mathbf{u}_1 + \lambda(\mathbf{u} - \mathbf{u}_1) \mid \lambda \geq 0\}$ with the boundary of P. The vertex $\mathbf{u}_2$ lies in the interior of some face which is spanned by at most d vertices of P. Hence $\mathbf{u} \in \operatorname{rel-int}([\mathbf{u}_1, \mathbf{u}_2])$ can be represented as a positive convex combination of $n + 1 \leq 2d + 1$ vertices of P including all vertices of S, let us say $\mathbf{u} = \sum_{i=0}^n \alpha_i' \mathbf{v}_i$ with $\sum_{i=0}^n \alpha_i' = 1$ and each $\alpha_i' > 0$.

Let $P' = \operatorname{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$. Choose a vertex $\mathbf{w} \in I_l(P')$ and a representation $\mathbf{w} = \sum_{i=0}^n \alpha_i \mathbf{v}_i$ with $\sum_{i=0}^n \alpha_i = 1$ maximizing $\min_{0 \leq i \leq n} \alpha_i$. Denote this maximum by $m(\mathbf{w}) > 0$. The argument of Theorem 2 shows that $m(\mathbf{w}) \geq \delta(n, \frac{8}{7}\, l)/8 \geq \delta(2d, \frac{8}{7}\, l)/8$.

The polytope P' can be represented as a projection of an n-simplex S_n such that $\mathbf{w}$ is the image of $\mathbf{v} \in \operatorname{int}(S_n)$ with $m_{S_n}(\mathbf{v}) = m(\mathbf{w})$. Now, it is easy to see that a linear mapping cannot increase the coefficient of asymmetry; hence

$$\mathrm{ca}(\mathbf{w}, P') \le \mathrm{ca}(\mathbf{v}, S_n) = \frac{1 - m(\mathbf{w})}{m(\mathbf{w})}.$$

It is known that $P \subset (-d)S + (d+1)\mathbf{s}$, where $\mathbf{s}$ is the centroid of S, see e.g. [3, Theorem 3]. By Lemma 3, we obtain

$$\mathrm{ca}(\mathbf{w}, P) \le d\,\mathrm{ca}(\mathbf{w}, P') + d - 1 \le d\,\frac{1 - m(\mathbf{w})}{m(\mathbf{w})} + d - 1 = \frac{d}{m(\mathbf{w})} - 1,$$

which give the required by (11). ∎

Remark. The bound (27) is much worse than (2); the reason is that we may have to approximate $2d$-tuples of numbers in Lemma 1. Unfortunately, we cannot guarantee that P' has much fewer than $2d+1$ vertices, as e.g. $P = \mathrm{conv}\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{d-1}, (k+1)l\mathbf{e}_d\}$ demonstrates. Perhaps, one can show that any such example cannot be extremal for our problem and thus improve on (27).

Remark. It should be possible to generalize Theorems 2 and 4 by proving the existence of a number $b = b(d, l, m) > 0$ such that any lattice polytope P contains m distinct points in $I_l(P)$ (provided $|I_l(P)| \ge m$) with coefficient of asymmetry of each being at least b. The idea of the proof is the following. If P is a simplex, take distinct $\mathbf{w}_1, \ldots, \mathbf{w}_m \in I_l(P)$ with with largest m_P's. Now each jump of $\mathbf{w}_i$ either does not increase $m_P(\mathbf{w}_i)$ or maps $\mathbf{w}_i$ into some other $\mathbf{w}_j$. We are done if we can show that if m_P is very small then there are at least m distinct jumps increasing it. The latter claim would be achieved by rewriting the proof of Lemma 1, so that in the conclusion we have at least m suitable $(n+1)$-tuples of integers. To extend the claim to general lattice polytopes, observe that m vertices in $I_l(P)$ can be represented each as a positive combination of $d+1$ vertices of a max-volume simplex and at most md other vertices of P and follow the argument of Theorem 4. We do not see any principal difficulties arising here, but it would take too much space to write the complete proof, so we restrict ourselves to this little observation only.

5 Volume of lattice simplices

For simplices we have a better method (than applying (8)) for bounding volume which appears in [2, Theorem 3.4] (see also [3, Lemma 2.3]). Let us reproduce this simple argument here.

Lemma 5 *Let $S = \operatorname{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_d\}$ be any simplex and let $\mathbf{w} \in I_l(S)$ have barycentric coordinates $(\alpha_0, \dots, \alpha_d)$. Then*

$$\operatorname{vol}(S) \le \frac{l^d}{d! \times \alpha_1 \times \alpha_2 \times \cdots \times \alpha_d} \, |I_l(S)|. \tag{28}$$

Proof. Consider the region

$$X = \{\mathbf{w} + \textstyle\sum_{i=1}^d \beta_i (\mathbf{v}_i - \mathbf{v}_0) \mid |\beta_i| \le \alpha_i \ 1 \le i \le d\}.$$

It is a centrally symmetric parallelepiped around the vertex $\mathbf{w} \in l\mathbb{Z}^d$ with volume $\operatorname{vol}(X) = d! \operatorname{vol}(S) \prod_{i=1}^d (2\alpha_i)$. We have to show that the volume of X cannot exceed $(2l)^d \, |I_l(S)|$. If this is not true, then X contains (besides $\mathbf{w}$) at least $|I_l(S)|$ pairs of vertices $\mathbf{w} \pm \mathbf{u} \in l\mathbb{Z}^d$ by Corput's theorem [8] and, clearly, at least one vertex of each such pair lies within $I_l(S)$, which is a contradiction. ∎

Now we can deduce the following result.

Theorem 6 *For any $k \ge 1$, the inequality* (10) *holds.*

Proof. Let $S \subset \mathbb{R}^n$ be lattice simplex with $|I_l(S)| = k$. By Theorem 2 there is $\mathbf{w} \in I_l(S)$ with $m_S(\mathbf{w}) \ge \gamma = (8l+7)^{-2^{d+1}}/8$. Assume $\alpha_0 \le \cdots \le \alpha_d$. Then it is easy to see that $\prod_{i=1}^d \alpha_i \ge \gamma^{d-1} (1 - \gamma d)$. The claim now follows from (28). ∎

6 $s(d, k, l)$ for small d and l

As we have already mentioned, $s(2, k, 1)$ was computed by Scott [7]. The simplex $S_{2,k,l}$ shows that $s(2, k, l) \ge l(l+1)^2(k+1)/2$. Upper bounds on $s(2, k, l)$ can be obtained by applying (28) to the lower bounds on $\beta(2, l)$

from Table 3. But even if we knew that (16) is sharp, the best upper bound on $s(2,k,l)$ that this method would give is $l^5 k/2 + O(l^4)k$, so there would still be an uncertainty about $s(2,k,l)$.

Also, an interesting problem is the determination of $s(3,k,1)$. The simplex $S_{3,k,1}$ shows that $s(3,k,1) \geq 6(k+1)$. Theorem 6 gives, already for such small d, very bad bounds. However, there is a very simple argument, following the lines of Section 3 and proving that

$$s(3,k,1) \leq \frac{29791}{2112} \cdot k < 14.106 \cdot k. \tag{29}$$

Given a lattice simplex $S \subset \mathbb{R}^3$, we can deduce as before that the barycentric coordinates $(\alpha_0, \ldots, \alpha_3)$ of a lattice vertex maximizing m_S satisfy $2\alpha_3 - 1 \leq \alpha_0$, $3\alpha_2 - 1 \leq \alpha_0$ and either $4\alpha_2 - 1 \leq \alpha_0$ or $4\alpha_3 - 2 \leq \alpha_0$. These inequalities do not guarantee yet that $\alpha_0 > 0$, but they guarantee that $\alpha_1 \geq \frac{2}{31}$ and, as it is routine to see, that $\alpha_1\alpha_2\alpha_3 \geq \frac{2}{31} \times \frac{11}{31} \times \frac{16}{31}$, which implies (29) by (28).

Of course, we could write more equations on the α's, but this method would not lead to the best possible bound. For example, the simplex $S_{3,1,1}$ shows that we cannot guarantee a vertex in $I_1(S)$ with $\alpha_1\alpha_2\alpha_3 > \frac{1}{2} \times \frac{1}{3} \times \frac{1}{12}$, so the best bound we would hope to obtain this way is $s(3,k,1) \leq 12k$ only.

7 Lawrence's Finiteness Theorem

A result of Lawrence [4, Lemma 5] implies that there is $\gamma = \gamma(d) > 0$ such that, for any lattice simplex $S = \mathrm{conv}\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$, the set $I_1(S)$ (if non-empty) contains a vertex $\mathbf{w}$ with $\alpha_0 \geq \gamma$, where $(\alpha_0, \ldots, \alpha_d)$ are the barycentric coordinates of $\mathbf{w}$. This directly follows from our Theorem 2 but unfortunately I could not find a simple argument giving the converse implication. And, in fact, the corresponding extremal functions are different: for example, $\beta(2,1) \leq 1/8$ while it is claimed in [4, p. 439] that we can take $\gamma(2) = 1/6$.

However, one can deduce from [4] that $\beta(d,1) > 0$ using the following simple modification of Lawrence's proof. For $i \in \mathbb{N}$ let $U_i = \{\mathbf{w} \in \Delta_d \mid \mathrm{ca}(\mathbf{w}, \Delta_d) < i\}$, where $\Delta_d = \mathrm{conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\}$. Clearly, $\cup_{i=d+1}^{\infty} U_i = \mathrm{int}(\Delta_d)$. By [4, Theorem 3] there exists i such that for any $\mathbf{w} \in \mathrm{int}(\Delta_d)$ there are $j \in \mathbb{N}$ and $\mathbf{u} \in \mathbb{Z}^d$ with with $j\mathbf{w} + \mathbf{u} \in U_i$. We claim that $\beta(d, 1) \geq \frac{1}{i+1}$. Indeed, let $S \subset \mathbb{R}^d$ be any lattice simplex and let $\mathbf{v} \in I_1(S)$. Choose any affine function $f : \mathbb{R}^d \to \mathbb{R}^d$ with $f(S) = \Delta_d$ and let $\mathbf{w} = f(\mathbf{v}) \in \mathrm{int}(\Delta_d)$. Given $\mathbf{w}$, choose the corresponding $j \in \mathbb{N}$ and $\mathbf{u} \in \mathbb{Z}^d$. It is easy to check that $\mathbf{v}' = f^{-1}(j\mathbf{w} + \mathbf{u})$ belongs to $I_1(S)$ and satisfies $m_S(\mathbf{v}') \geq \frac{1}{i+1}$.

But as it has already been remarked, Lawrence's argument does not give any explicit bound.

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