# Enumeration of Labelled $(k, m)$-Trees


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**Abstract**

A $k$-graph is called a $(k, m)$*-tree* if it can be obtained from a single edge by consecutively adding edges so that every new edge contains $k - m$ new vertices while its remaining $m$ vertices are covered by an already existing edge. We prove that there are

$$\frac{(e(k-m)+m)!(e\binom{k}{m}-e+1)^{e-2}}{e!m!\,((k-m)!)^e}$$

distinct vertex labelled $(k, m)$-trees with $e$ edges.


The notion of a tree and its different extensions to $k$*-graphs*, that is, $k$-uniform set systems, play an important role in discrete mathematics and computer science. We will dwell upon the following, rather general, definition suggested independently by Dewdney [5] and Beineke and Pippert [2].

Let us agree that the vertex set is $[n] = \{1, \ldots, n\}$. Fix the *edge size* $k$ and the *overlap size* $m$, $0 \le m \le k - 1$. We refer to $k$-subsets and $m$-subsets of $[n]$ as *edges* and *laps* respectively. A non-empty $k$-graph without isolated vertices is called a $(k, m)$*-tree* if we can order its edges, say $E_1, \ldots, E_e$, so


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that for every $i$, $2 \leq i \leq e$, there is $i'$, $1 \leq i' < i$, such that $|E_i \cap E_{i'}| = m$ and $(E_i \setminus E_{i'}) \cap \left(\cup_{j=1}^{i-1} E_j\right) = \emptyset$. In other words, we start with a single edge and can consecutively affix a new edge along an $m$-subset of an existing edge.

Thus, a $(k, m)$-tree with $e$ edges has $n = e(k - m) + m$ vertices and its edges cover $f = e\left(\binom{k}{m} - 1\right) + 1$ laps. For example, a $(k, 0)$-tree consists of disjoint edges.

The problem of counting $(m + 1, m)$-trees which are known in the literature as *$m$-trees*, received great attention and was completely settled by Beineke and Pippert [1] and Moon [9]. This extends celebrated Cayley's theorem [3] as, clearly, 1-trees correspond to usual (Cayley) trees. Later, different bijective proofs for $m$-trees appeared as well, see [11, 7, 8, 6, 4].

In this paper we enumerate $(k, m)$-trees. It is not surprising that many of the above counting techniques are applicable here as is, for example, Foata's bijection [7] (details are available from the author [10]). Also, as observed by the referee, our formula can be obtained via Chen's method [4]. We decided to present here an inductive proof which is the shortest one, perhaps.

Let $T_{km}(e)$ denote the number of $(k, m)$-trees on $[n]$ with $e$ edges, $n = e(k - m) + m$, and let $R_{km}(e)$ count the trees *rooted* at the lap $[m]$, that is, those trees for which $[m]$ is covered by some edge.

**Theorem 1** *Given integers $k$, $m$, $e$ with $0 \leq m \leq k - 1$ and $e \geq 1$, let $n = e(k - m) + m$, $l = \binom{k}{m}$ and $f = e(l - 1) + 1$. Then the number of different $(k, m)$-trees on $[n]$ equals*

$$T_{km}(e) = \frac{n! f^{e-2}}{e! m! \left((k - m)!\right)^e}. \tag{1}$$

**Proof.** Like in Beineke and Pippert [1], to prove the theorem, we write down a recurrence relation for $T_{km}(e)$ and then verify that (1) does satisfy the relation. Let us agree that $T_{km}(0) = R_{km}(0) = 1$.

Counting in two different ways the number of pairs $(H, L)$, where $H$ is a $(k, m)$-tree on $[n]$ rooted at $L \in [n]^{(m)}$, we obtain

$$\binom{n}{m} R_{km}(e) = f \cdot T_{km}(e). \tag{2}$$

Next, consider the following method for constructing trees. Select an edge $E \in [n]^{(k)}$ and label by $L_1, \ldots, L_l$ the laps of $E$. Represent $e-1$ as a sum of $l$ non-negative integers, $e-1 = e_1 + \ldots + e_l$. Partition $[n] \setminus E$ into sets $X_1, \ldots, X_l$ of sizes $e_1(k-m), \ldots, e_l(k-m)$ respectively. On each $L_i \cup X_i$ build a $(k,m)$-tree $H_i$ rooted at $L_i$, $i \in [l]$. Clearly, the union of all $H_i$'s plus the edge $E$ forms a $(k,m)$-tree and every such tree is obtained exactly $e$ times. Therefore, by (2), we obtain

$$\begin{aligned} eT_{km}(e) &= \binom{n}{k} \sum_{\mathbf{e}} \frac{(n-k)!}{(e_1(k-m))! \ldots (e_l(k-m))!} \prod_{i=1}^{l} R_{km}(e_i) \\ &= \frac{n!}{k!} \sum_{\mathbf{e}} \prod_{i=1}^{l} \frac{m!(e_i(l-1)+1)T_{km}(e_i)}{(e_i(k-m)+m)!}, \end{aligned} \qquad (3)$$

where $\sum_{\mathbf{e}}$ denotes the summation over all representations $e-1 = e_1 + \ldots + e_l$ with non-negative integer summands.

Clearly, formula (1) gives correct values for $e = 0$. Also, the substitution of (1) into the both sides of (3) gives (after routine cancellations)

$$l(e(l-1)+1)^{e-2} = \sum_{\mathbf{e}} \frac{(e-1)!}{e_1! \ldots e_l!} \prod_{i=1}^{l} (e_i(l-1)+1)^{e_i-1}.$$

The last identity (in slightly different notation) was established by Beineke and Pippert [1, Lemma 2], which proves our theorem by induction. □

**Corollary 1** *The number of labelled $m$-trees on $n$ vertices, $n > m \geq 1$, is $T_{m+1,m}(n-m) = \binom{n}{m}(mn - m^2 + 1)^{n-m-2}$.* □

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