Weakly Saturated Hypergraphs and Exterior Algebra

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January 14, 2000

Abstract

Given an r-graph F, an r-graph G is called weakly F-saturated if the edges missing from G can be added, one at a time, in some order, each extra edge creating a new copy of F. Let $\text{w-sat}(n, F)$ be the minimal size of a weakly F-saturated graph of order n. We compute the w-sat-function for a wide class of r-graphs called *pyramids*: these include many examples for which the w-sat-function was known, as well as many new examples, such as the computation of $\text{w-sat}(n, K_s + \overline{K_t})$ and the proof of a conjecture of Tuza.

Our main technique *G-proof*, developed from ideas of Kalai, is based on matroids derived from exterior algebra. We prove that if some graphs admit G-proof then so do 'cones' and 'joins' of such graphs, so that the w-sat-function can be computed for many graphs that are built up from certain elementary graphs by these operations.

1 Introduction

An *r-graph* F is a pair $(V(F), E(F))$ (*vertices* and *edges*), where

$$E(F) \subset V(F)^{(r)} = \{A \subset V(F) : |A| = r\}.$$

By analogy with the case $r = 2$, the *size* of F is $e(F) = |E(F)|$, the *order* is $v(G) = |V(F)|$, and in the obvious way we define the notions of an *isomorphism*, a *subgraph*, the *complementary graph* $\overline{F}$, etc.

Given a family $\mathcal{F}$ of 'forbidden' r-graphs, an r-graph G of order n is called *weakly $\mathcal{F}$-saturated* (denoted by $G \in \text{w-SAT}(n, \mathcal{F})$) if we can consecutively add all missing edges to G so that every time we add an edge a new *F-subgraph* (a subgraph

*Supported by a Senior Rouse Ball Studentship, Trinity College, Cambridge, UK.

isomorphic to F), some $F \in \mathcal{F}$, appears. We call the corresponding ordering of $E(\overline{G})$ *$\mathcal{F}$-proper*. The principal problem we deal with, is the determination of

$$\text{w-sat}(n, \mathcal{F}) = \min\{e(G) : G \in \text{w-SAT}(n, \mathcal{F})\}.$$

Note that we do not require that G is $\mathcal{F}$-free as this does not affect w-sat$(n, \mathcal{F})$. If we have a single forbidden graph F, we write w-SAT(n, F) for w-SAT$(n, \{F\})$, etc. The related notion of (strong) saturation (studied in e.g. [2, 9, 5, 13]) is not considered in this work.

The w-sat-function was introduced by Bollobás [3]. Its computation seems a difficult task and few results are known. Usually, given $\mathcal{F}$, it is easy to find a right example of $G \in$ w-SAT$(n, \mathcal{F})$ but it is hard to prove that G is indeed minimum. There does not seem to be any systematic approach to the latter task.

Here we present *G-proof*, a sufficient condition for $G \in$ w-SAT$(n, \mathcal{F})$ being extremal. This is a deterministic procedure which utilises matroids constructed via exterior algebra. The links with matroid theory are not surprising: loosely speaking, an $\mathcal{F}$-proper addition of edges corresponds to closure and the notion of a minimum weakly saturated graph resembles that of a base. Applying G-proof and other related methods (g/g$'$/etc.-proofs), we obtain various results as follows.

Let sequences $\mathbf{r} = (r_1, \ldots, r_t)$ of non-negative integers and $S_1, \ldots, S_t$ of disjoint sets of sizes $\mathbf{s} = (s_1, \ldots, s_t)$ be given. We define $[t] = \{1, ..., t\}$ and, for $I \subset [t]$, we use shortcuts such as $r_I = \sum_{i \in I} r_i$ and $S_I = \cup_{i \in I} S_i$; also we assume $r_0 = 0$, $S_0 = \emptyset$, etc. Then the *pyramid* $P = P(\mathbf{s}, \mathbf{r})$ is the r-graph, $r = r_{[t]}$, on $S = S_{[t]}$ such that $E \in S^{(r)}$ is an edge of P if and only if for every $i \in [t]$ we have $|E \cap S_{[i]}| \geq r_{[i]}$.

Perhaps our most important result is the exact computation (via G-proof) of the w-sat-function for pyramids.

Theorem 1 *Suppose we are given two non-empty sequences $\mathbf{s} = (s_1, \ldots, s_t)$ and $\mathbf{r} = (r_1, \ldots, r_t)$ of integers such that $s_i \geq r_i \geq 1$ for $i \in [t]$. Then*

$$\text{w-sat}(n, P(\mathbf{s}; \mathbf{r})) = \sum_{\mathbf{r}'} \binom{n - s_{[t]} + r_t}{r'_{t+1}} \prod_{i \in [t]} \binom{s_i + r_{i-1} - r_i}{r'_i}, \quad n \geq s_{[t]}, \qquad (1)$$

where the summation is taken over all sequences of non-negative integers $\mathbf{r}' = (r'_1, \ldots, r'_{t+1})$ such that $r'_{[t+1]} = r_{[t]}$ and, for some $i \in [t]$, $r'_{[i]} > r_{[i-1]}$.

The notion of pyramid is quite general. For example, for $t = 1$ we have $P(s; r) = K_s^r$ and we obtain the formula

$$\text{w-sat}(n, K_s^r) = \binom{n}{r} - \binom{n - s + r}{r}, \quad n \geq s \geq r \geq 1 \qquad (2)$$

conjectured by Bollobás [3] and proved by Lovász [10], Frankl [6], and Kalai [7]. On the other hand, we obtain new results even for 2-graphs: $P(s, t; 1, 1) = K_s^2 + \overline{K_t^2}$ and we have

$$\text{w-sat}(n, K_s^2 + \overline{K_t^2}) = (s - 1)n - \binom{s}{2} + \binom{t}{2}, \quad n \geq s + t, \ s, t \geq 1.$$

Also, $P(r-l+1, l; r-l+1, l-1)$ is the only graph in $\mathcal{H}_r(r+1, l)$, where the *uniform family* $\mathcal{H}_r(m, l)$ consists all r-graphs on m vertices with l edges. Hence, (1) implies the following result conjectured by Tuza [14, Conjecture 7].

$$\text{w-sat}(n, \mathcal{H}_r(r+1, l)) = \binom{n-r+l-2}{l-2}, \quad n \geq r+1 \geq l \geq 2.$$

The case $l = 3$ of Tuza's conjecture was proved by Erdős, Füredi and Tuza [5].

An advantage of G-proof is that it might help in guessing answers. For example, a correct example of $G \in \text{w-sat}(n, P(\mathbf{s}; \mathbf{r}))$ was obtained by the author by guessing first the more transparent construction (9) which plays the key role in the proof.

The *cone* $\text{cn}(F)$ of an r-graph F is obtained by adding to F a new vertex v and all r-edges containing v; $\text{cn}(\mathcal{F}) = \{\text{cn}(F) : F \in \mathcal{F}\}$ for a family $\mathcal{F}$ of graphs. Our more general Theorem 14 implies that the pair $\text{cn}(G) \in \text{w-SAT}(n, \text{cn}(\mathcal{F}))$ admits G/g/g′-proof provided so does the pair $G \in \text{w-SAT}(n, \mathcal{F})$ and any $r-1$ vertices in any $F \in \mathcal{F}$ are covered by at least one edge. Of course, this result is useful only if we know many graphs admitting G/g/g′-proof. Various examples of such graph are indicated in this work. For 2-graphs, they include: complete graphs, stars, odd cycles, initial colex-segments of $[n]^{(2)}$, disjoint edges (more generally, almost every forest or tree). Therefore, we are able to compute the w-sat-function for $\text{cn}^l(F) = K_l^2 + F$, where F is any of these graphs. Also, (2) can be easily deduced from our cone-result by observing that $\text{cn}(K_s^r) = K_{s+1}^r$.

In Section 5 we define the join-operator and prove that it 'preserves' G/g/r-proofs. (See Theorem 17 for the precise statement.) For example, Alon's [1] computation of the w-sat-function for joins of complete graphs (another proof is presented by Yu [15]) is a special instance of our result.

Unfortunately, G/g/g′-proofs are not easy to handle and there are many concrete examples of extremal graphs not admitting any of them. Besides, G/g/g′-proofs do not provide the immediate characterization of all minimum graphs. But we hope that they will lead to further results.

2 Proof Techniques

In Subsection 2.2 we associate with every r-graph G its gross matroid. As this definition relies on exterior algebra, we provide some background in Subsection 2.1; for a comprehensive introduction to the topic the reader can consult Marcus [11], for example. Then, in Subsection 2.3, we describe how we apply these notions to w-sat-type problems.

2.1 Exterior Algebra

Let V be an n-dimensional real vector space with basis $\mathbf{e} = \{e_1, \dots, e_n\}$. Its *exterior algebra* $\bigwedge V$ is a 2^n-dimensional vector space with the formal basis $(e_A)_{A \subset [n]}$. (We identify e_i with $e_{\{i\}}$ and $e_\emptyset$ with the scalar $1 \in \mathbb{R}$.) It comes equipped with the associative bilinear $\wedge$-product which is completely determined by $e_i \wedge e_j = -e_j \wedge e_i$, $i, j \in [n]$, and

$$e_{v_1} \wedge \ldots \wedge e_{v_k} = e_{\{v_1, \ldots, v_k\}}, \quad 1 \le v_1 < \ldots < v_k \le n.$$

Let $(e^*_A)_{A \subset [n]}$ be the dual basis of $(e_A)_{A \subset [n]}$. We naturally identify $\bigwedge V^*$ and $(\bigwedge V)^*$ so that $e^*_{v_1} \wedge \ldots \wedge e^*_{v_k}$ corresponds to $e^*_{\{v_1, \ldots, v_k\}}$, $1 \le v_1 < \ldots < v_k \le n$.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be another basis of V; in the obvious way we define f_A, f^*_A, etc. By $M = (\alpha_{ij})_{i,j \in [n]}$ we denote the $n \times n$-matrix satisfying $\mathbf{f}^* = M\mathbf{e}^*$, that is,

$$f^*_i = \alpha_{i1} e^*_1 + \ldots + \alpha_{in} e^*_n, \qquad i \in [n].$$

Assume that $\mathbf{f}$ is in the *generic* position with respect to $\mathbf{e}$, that is, the entries of M are n^2 transcendentals algebraically independent over the rationals. Any equation we will consider can be reduced to the form $P = 0$ for some polynomial P in the α's with integer coefficients and we agree that the statement is true if and only if P is the zero polynomial.

Let $\bigwedge^i V$ be the subspace of $\bigwedge V$ spanned by $(e_A)_{A \in [n]^{(i)}}$. For $h \in \bigwedge V$ its *support* is defined by $\mathrm{supp}(h) = \{A \subset [n] : e^*_A(h) \ne 0\}$. If we take the support in the basis $\mathbf{f}$, we emphasize this by adding a subscript: $\mathrm{supp}_{\mathbf{f}}(h) = \{A \subset [n] : f^*_A(h) \ne 0\}$. For $g^* \in \bigwedge V^*$, $h \in \bigwedge V$ denote $\langle g^*, h \rangle = g^*(h)$ and define the *left interior product* $g^* \llcorner h \in \bigwedge V$ by

$$\langle u^*, g^* \llcorner h \rangle = \langle u^* \wedge g^*, h \rangle, \quad \text{for all } u^* \in \textstyle\bigwedge V^*.$$

One can easily check that $\llcorner$ is a bilinear function such that $u^* \llcorner (g^* \llcorner h) = (u^* \wedge g^*) \llcorner h$ and, for the basis vectors, we have:

$$e^*_A \llcorner e_B = \begin{cases} \pm e_{B \setminus A}, & \text{if } A \subset B, \\ 0, & \text{if } A \not\subset B. \end{cases}$$

(The actual signs of ± 1-coefficients do not interest us.) Note that the cancellation $(g^* \wedge e^*_A) \llcorner (h \wedge e_A) = g^* \llcorner h$, which is not generally correct, can be applied if, for example, each $B \in \mathrm{supp}(h)$ is disjoint from A.

2.2 Construction

Let us describe how to construct the *gross matroid* $\mathcal{G}_G$ of an r-graph G of order n. This construction is not new; e.g. Kalai [8] used it to construct symmetric matroids with a given growth polynomial. Also, in the partial case $G = P(k, n-k; 1, 1)$, the matroid $\mathcal{G}_G$ is exactly Kalai's [7] *k-hyperconnectivity matroid* on $[n]^{(2)}$ which was applied to w-sat-type problems. These two papers by Kalai were the starting points of our research on gross matroids.

Let $V(G) = [n]$ and let $Z \subset \bigwedge^r V$ be defined by the following linear relations:

$$Z = \{h \in \textstyle\bigwedge^r V : f^*_E \llcorner h = 0 \text{ for all } E \in E(G)\}. \tag{3}$$

Clearly, $\dim Z = \binom{n}{r} - e(G)$ and, in fact, Z is spanned by $\{f_E : E \in E(\overline{G})\}$.

We define the *gross matroid* $\mathcal{G}_G$ on $[n]^{(r)}$ so that an r-graph F on $[n]$ is dependent if, for some coefficients c_E (not all zero), we have $\sum_{E\in E(F)} c_E e_E \in Z$, that is,

$$\sum_{E\in E(F)} c_E f_D^* \llcorner e_E = 0, \qquad D \in E(G). \tag{4}$$

By $M(G,F)$ we denote the $e(G)\times e(F)$-matrix corresponding to (4). The columns of $M(G,[n]^{(r)})$ provide a representation of $\mathcal{G}_G$. Note that the matroid $\mathcal{G}_G$ does not depend on the choice of generic $\mathbf{f}$. Also, $\mathcal{G}_G$ is a *symmetric* matroid, that is, for any permutation $\sigma : [n] \to [n]$, $A \subset [n]^{(r)}$ is $\mathcal{G}_G$-independent if and only if $\sigma'(A)$ is, where σ' is the induced action on $[n]^{(r)}$. Therefore, we can apply the notion of $\mathcal{G}_G$-dependence to an r-graph F with any vertex set. (If $v(F) > v(G)$, we add isolated vertices to G.)

The rank of $\mathcal{G}_G$ is $\mathrm{codim}(Z) = e(G)$. It is easy to show that G is a base of $\mathcal{G}_G$. Indeed, the determinant of $M(G,G)$ is a polynomial in the α's which assumes value 1 when M (and then $M(G,G)$) is the identity matrix. Therefore, the determinant is non-zero for generic M, which proves the claim.

2.3 Our Approach

The following observation, due to Kalai [7], is crucial to our work. Suppose that $\mathcal{F}$ is an r-graph family and $\mathcal{M}$ is a matroid on $[n]^{(r)}$ such that, for every $F\in\mathcal{F}$ and for every embedding $V(F)\subset[n]$, the set $E(F)\subset[n]^{(r)}$ is an $\mathcal{M}$*-cycle*, that is, every edge $E\in E(F)$ is dependent on $E(F)\setminus\{E\}$. Let $G\in$ w-SAT$(n,\mathcal{F})$ and let $E_1,\dots,E_k$ be an $\mathcal{F}$-proper ordering of $E(\overline{G})$. Then, for every $i\in[k]$, there is an F-subgraph of $G_i = G + E_1 \ldots + E_i$ which contains E_i, some $F\in\mathcal{F}$. Thus, E_i lies in the $\mathcal{M}$-closure of G_{i-1}, which inductively implies that G spans $[n]^{(r)}$ in $\mathcal{M}$. Hence,

$$\text{w-sat}(n,\mathcal{F}) \geq R_{\mathcal{M}}([n]^{(r)}). \tag{5}$$

In this case we say that we can *m-prove* the inequality (5). If $\mathcal{M}$ is a gross matroid or a representable matroid, then (5) is said to be *g-proved* or *r-proved* correspondingly. If the lower bound in (5) is sharp, then we say that $\mathcal{F}$ admits an *m-proof* for n. In the obvious way we define a *g-proof* and an *r-proof*.

Furthermore, define $D_{\mathcal{M}}(\mathcal{F}) = \min\{D_{\mathcal{M}}(F) : F\in\mathcal{F}\}$, where $D_{\mathcal{M}}(F) = \min_{F\subset[n]}(e(F) - R_{\mathcal{M}}(E(F)))$. The first edge E_1 added to G creates some forbidden $F\subset[n]$; clearly, $E(F)\setminus\{E\}\subset E(G)$. Therefore, there are $D_{\mathcal{M}}(F)-1$ edges in G which are dependent on the remaining edges and we can improve (5) as follows.

$$\text{w-sat}(n,\mathcal{F}) \geq R_{\mathcal{M}}([n]^{(r)}) + D_{\mathcal{M}}(\mathcal{F}) - 1. \tag{6}$$

We say that (6) is *m'-proved.* If $\mathcal{M}$ is a gross or representable matroid, then we respectively *g'-prove* or *r'-prove* (6). If (6) is sharp, then we obtain an *$m'/g'/r'$-proof* correspondingly.

As we have already mentioned, it is usually easy to find a right example of $G\in$ w-SAT$(n,\mathcal{F})$ but hard to prove its extremality. Even if there exists an m-proof, it is not obvious at all how to search for a suitable matroid. However, in our approach we suggest to consider the gross matroid of G as a candidate for $\mathcal{M}$. If each graph in $\mathcal{F}$ is a $\mathcal{G}_G$-cycle, then we immediately conclude that G is extremal. In this case say that the pair $(\mathcal{F}, G)$ admits *G-proof*. Hence, G-proof can be viewed as a sufficient criterion for $G \in \text{w-SAT}(n, F)$ to be of the minimal size.

It is easy to see that an r-graph F is a $\mathcal{G}_G$-cycle if and only if there is $h \in Z$ with $\text{supp}(h) = E(F)$. To verify this condition we have to find a solution $(c_E)_{E\in E(F)}$ with all entries non-zero of the system (4).

Remark. Note that if $\mathbf{f}$ coincides with $\mathbf{e}$, then the relation 'F is independent in $\mathcal{G}_G$' is equivalent to $E(F) \subset E(G)$,' so we are tempted to call the former relation (for generic $\mathbf{f}$) the gross version of '$F \subset G$.' Then $G \in \text{w-SAT}(n, \mathcal{F})$ admits G-proof if and only if the gross version of the following holds:

$$\text{“}G \text{ is } \mathcal{F}\text{-free, but } G + E \text{ is not for any new edge } E \in E(\overline{G}).\text{”} \tag{7}$$

This poses a few intriguing questions: why is predicate (7) good for tackling the w-sat-problem? can we obtain any progress on other combinatorial questions if we take some other predicate? if yes, how can we find one?

Let us prove one trivial lemma which, when combined with the results of Sections 4 and 5, has non-trivial consequences.

Lemma 2 *Let $K = lK_r^r$ be the union of l disjoint r-edges. Then $\mathcal{G}_K$ is the* uniform matroid *of rank l, that is, an r-graph F is independent in $\mathcal{G}_K$ if and only if $e(F) \le l$.*

In particular, for any family $\mathcal{F}$ of r-graphs and for any n with $\binom{n}{r} \ge l$, we can g-prove that $\text{w-sat}(n, \mathcal{F}) \ge l$*, where* $l = \min\{e(F) : F \in \mathcal{F}\} - 1$.

Proof. We show, by induction on l, that any r-graph H of size l is $\mathcal{G}_K$-independent. Assume that $E = [r]$ is an edge in both these graphs. One can see that

$$\det(M(K, H)) = \pm\alpha_{11} \dots \alpha_{rr} \det(M(K', H')) + (\text{other terms}),$$

where H' and K' are obtained respectively from H and K by removing E and none of the 'other terms' contains $\alpha_{11} \dots \alpha_{rr}$ as a factor. By induction, we conclude that $\det(M(K, H)) \neq 0$, and all claims follow. $\square$

3 Specific Classes

Here we obtain various results for some particular forbidden families.

3.1 Pyramids

Here we calculate $\text{w-sat}(n, P(\mathbf{s}, \mathbf{r}))$ by showing that pyramids admit G-proof. Note that we obtain the exact answer for *all feasible* values of the parameters n, $\mathbf{r}$ and $\mathbf{s}$.

Let $P = P(\mathbf{s}, \mathbf{r})$. Assume $s_i \ge r_i \ge 1$ for $i \in [t]$ as it is not hard to see that any non-empty pyramid has a such representation.

Let us, for any $n \geq s = s_{[t]}$, provide a construction of $G \in \text{w-SAT}(n, P)$. Partition $[n] = A_1 \cup \ldots \cup A_{t+1}$ so that $a_i = |A_i| = s_i + r_{i-1} - r_i$, $i \in [t]$; thus

$$a_{t+1} = |A_{t+1}| = n - \sum_{i=1}^{t}(s_i + r_{i-1} - r_i) = n - s + r_t.$$

We also assume that our partition is *consecutive*, that is, in $[n]$, any element of A_i comes before any element of A_j whenever $i < j$. Let $E \in [n]^{(r)}$ be an edge of G if and only if for some $i \in [t]$ we have $|E \cap A_{[i]}| > r_{[i-1]}$. Equivalently, the complement of G is isomorphic to $P(a_{t+1}, \ldots, a_1; r_t, \ldots, r_1, 0)$, so, for example, any r-tuple intersecting A_1 is in $E(G)$.

Lemma 3 *$G \in \text{w-SAT}(n, P)$.*

Proof. Order the missing edges in any way so that the sequences

$$\left(|A_{[1]} \cap E|, \ldots, |A_{[t+1]} \cap E|\right), \quad E \in E(\overline{G}),$$

are non-increasing in the lexicographic order. (Thus, we start with $(0, r_1, \ldots, r_t)$ and end with $(0, \ldots, 0, r)$.) Let us show that this ordering is P-proper. Consider the moment when we add some edge $E \in E(\overline{G})$. Let $E_i = E \cap A_{i+1}$, $i \in [t]$. Also, let $E = R_1 \cup \ldots \cup R_t$ and $[n] \setminus E = T_1 \cup \ldots \cup T_{t+1}$ be the consecutive partitions with $|R_i| = r_i$ and $|T_i| = s_i - r_i$, $i \in [t]$.

Let us show that $E_{[i]} \subset R_{[i]}$ and $T_{[i]} \subset A_{[i]} \setminus E_{[i-1]}$, $i \in [t]$. As all partitions in question are consecutive, it is enough to verify the sizes. By the definition of G we have $|E_{[i]}| = |E \cap A_{[i+1]}| \leq r_{[i]}$. Also,

$$|A_{[i]} \setminus E_{[i-1]}| \geq |A_{[i]}| - r_{[i-1]} = \sum_{j=1}^{i}(s_j + r_{j-1} - r_j) - r_{[i-1]} = |T_{[i]}|,$$

and the claim follows.

Let $S_i = T_i \cup R_i$, $i \in [t]$. We claim that E creates a forbidden subgraph P on the set $S = S_{[t]}$. For every $i \in [t]$ we have $|E \cap S_i| = |R_i| = r_i$, so $E \in E(P)$.

Suppose, on the contrary, that there exists $D \in E(P)$ coming after E. Let us show by induction on i that for every $i \in [0, t]$ we have

$$D \cap S_{[i]} = E \cap S_{[i]} \quad \text{and} \quad D \cap A_{[i+1]} = E \cap A_{[i+1]}, \tag{8}$$

which would be a contradiction to the assumption $D \neq E$. As $D, E \in E(\overline{G})$ are disjoint from A_1, the claim is true for $i = 0$. Let $i \in [t]$. As $T_{[i]} \subset A_{[i]}$, we conclude, by the inductive assumption, that $D \cap T_{[i]} = E \cap T_{[i]} = \emptyset$. As $S_{[i]} = T_{[i]} \cup R_{[i]}$, we have $D \cap S_{[i]} \subset R_{[i]}$. On the other hand, $D \in E(P)$, so $|D \cap S_{[i]}| \geq r_{[i]}$, which implies

$$D \cap S_{[i]} = R_{[i]} = E \cap S_{[i]},$$

and the first part of (8) is proved. Now,

$$D \cap A_{[i+1]} \supset R_{[i]} \cap A_{[i+1]} \supset E_{[i]} \cap A_{[i+1]}.$$

By induction, $D \cap A_{[i]} = E \cap A_{[i]}$ and, as D was added later than E, we must have $|D \cap A_{[i+1]}| \leq |E \cap A_{[i+1]}|$, which proves (8) completely. □

Theorem 4 *The pair (P, G) admits G-proof.*

Proof. We have to show that P is a $\mathcal{G}_G$-cycle. Let us consider

$$h = h_1 \wedge \ldots \wedge h_t, \quad \text{where} \;\; h_i = f^*_{A_{[i]}} \llcorner e_{S_{[i]}} \in \bigwedge^{r_i} V, \;\; i \in [t]. \tag{9}$$

Each $E \in \mathrm{supp}(h)$ is of the form $E'_1 \cup \ldots \cup E'_t$, for some $E'_i \in \mathrm{supp}(h_i)$, $i \in [t]$. Clearly, $|E'_i| = r_i$ and $E'_i \subset S_{[i]}$. Therefore, $|E \cap S_{[i]}| \geq |E'_{[i]}| = r_{[i]}$, so $\mathrm{supp}(h) \subset E(P)$. Similarly, $\mathrm{supp}_{\mathbf{f}}(h_i)$ lives within $A_{[i+1,t+1]}$, $i \in [t]$, which implies that $\mathrm{supp}_{\mathbf{f}}(h) \subset E(\overline{G})$.

So, to prove the theorem, it is enough to show that for any $E \in E(P)$ we have $P_E = \langle e^*_E, h \rangle \neq 0$. To do so, we can assume that S is an initial segment in $[n]$ and every element of S_i comes before every element of S_j whenever $i < j$. Furthermore, we can assume that $E_i = E \cap S_i$ is a final segment of S_i. Note that $A_{[i]} \subset S_{[i]} \subset A_{[i+1]}$ and $R_i = S_{[i]} \setminus A_{[i]}$ consists of the last r_i elements of S_i, $i \in [t]$. Clearly, $|E| = |R|$, where $R = R_{[t]}$, so let $g : E \setminus R \to R \setminus E$ be the order-preserving bijection.

As P_E is a polynomial in the α's, to show that $P_E \neq 0$, it is enough to demonstrate a particular example of the α's (or $\mathbf{f}^*$) such that $P_E \neq 0$. Define

$$f^*_x = \begin{cases} e^*_x + e^*_{g(x)}, & x \in E \setminus R, \\ e^*_x, & \text{otherwise.} \end{cases} \tag{10}$$

Let $i \in [t]$. Denote $W_i = A_{[i]} \setminus (E \setminus R)$ and

$$\begin{aligned} X_i &= \{x \in A_{[i]} \setminus W_i : g(x) \in A_{[i]}\}, \\ Y_i &= \{x \in A_{[i]} \setminus W_i : g(x) \notin S_{[i]}\}, \\ Z_i &= \{x \in A_{[i]} \setminus W_i : g(x) \in S_{[i]} \setminus A_{[i]}\}. \end{aligned}$$

As $A_{[i]} \subset S_{[i]}$, we have a partition $A_{[i]} = W_i \cup X_i \cup Y_i \cup Z_i$. As $f^*_x = e^*_x$ for $x \in W_i$ and $g(x) \in W_i$ for $x \in X_i$, we have

$$\begin{aligned} f^*_{A_{[i]}} &= \pm e^*_{W_i} \wedge f^*_{X_i \cup Y_i \cup Z_i} = \pm e^*_{W_i \setminus g(X_i)} \wedge \left(\textstyle\bigwedge_{x \in X_i} (e^*_x + e^*_{g(x)}) \wedge e^*_{g(x)}\right) \wedge f^*_{Y_i \cup Z_i} \\ &= \pm e^*_{W_i \setminus g(X_i)} \wedge \left(\textstyle\bigwedge_{x \in X_i} e^*_x \wedge e^*_{g(x)}\right) \wedge f^*_{Y_i \cup Z_i} = \pm e^*_{W_i \cup X_i} \wedge f^*_{Y_i \cup Z_i}. \end{aligned}$$

Next, $g(Y_i) \cap S_{[i]} = \emptyset$ and $g(Z_i) \subset S_{[i]} \setminus A_{[i]} = R_i$. Hence

$$\begin{aligned} h_i &= \pm \left(e^*_{W_i \cup X_i} \wedge \left(\textstyle\bigwedge_{x \in Y_i} (e^*_x + e^*_{g(x)})\right) \wedge f^*_{Z_i}\right) \llcorner e_{S_{[i]}} \\ &= \pm \left(e^*_{W_i \cup X_i \cup Y_i} \wedge f^*_{Z_i}\right) \llcorner e_{S_{[i]}} = \pm f^*_{Z_i} \llcorner e_{Z_i \cup R_i}. \end{aligned}$$

For $i \in [t]$ we have $|E_{[i-1]}| \geq |R_{[i-1]}|$ and one of E_i and R_i is a subset of the other, so, for each $x \in E_i \setminus R$, $g(x)$ lies in $R_j = S_{[j]} \setminus A_{[j]}$ and $x \in Z_j$ for some $j \in [i+1, t]$. Therefore, $Z_{[t]} = E \setminus R$.

When we compute $P_E = \pm \langle e^*_E, \wedge_{i \in [t]} (f^*_{Z_i} \llcorner e_{Z_i \cup R_i}) \rangle$ by expanding further each h_i in the $\mathbf{e}$-basis, we obtain h as a sum of terms each of the form e_D, for some $D \in [n]^{(r)}$. By definition, $\langle e^*_E, e_D \rangle = 0$ unless $E = D$. Consider some $x \in Z_i$.

As $x \notin R$ and $Z_1, \ldots, Z_t$ are disjoint, no element of $\mathrm{supp}(h_j)$ can contain x unless $j = i$. Computing h_i, we have

$$h_i = \pm(f^*_{Z_i \setminus \{x\}} \wedge e^*_{g(x)}) \llcorner e_{Z_i \cup R_i} \pm (f^*_{Z_i \setminus \{x\}} \wedge e^*_x) \llcorner e_{Z_i \cup R_i},$$

and no element in the **e**-support of the second summand can contain x. Thus we can harmlessly replace f^*_x by $e^*_{g(x)}$. (Clearly, this does not affect h_j for $j \neq i$.) Now,

$$P_E = \pm\langle e^*_E, \wedge_{i\in[t]}(e^*_{g(Z_i)} \llcorner e_{Z_i \cup R_i})\rangle = \pm\langle e^*_E, e_{Z_{[t]} \cup R_{[t]} \setminus g(Z_{[t]})}\rangle = \pm\langle e^*_E, e_E\rangle = \pm 1.$$

Thus P_E is non-zero and the theorem is proved. $\square$

Now the promised Theorem 1 clearly follows.

3.2 Uniform Families

Let us consider uniform families studied by e.g. Erdős, Füredi and Tuza [5]. By definition, $G \in \text{w-SAT}(n, \mathcal{H}_r(m,l))$ if we can consecutively add all missing edges to G so that every time we create a new subgraph with at most m vertices and at least l edges. In [12] we present a general construction of a weakly $\mathcal{H}_r(m,l)$-saturated graph of order n, which we conjecture to be extremal, and prove some partial cases of our conjecture. Here we obtain further results confirming it: we compute $\text{w-sat}(n, \mathcal{H}_r(m,l))$ asymptotically for $l > \binom{m}{r} - m + r$ and exactly for $r = 2$.

First, we need one simple preliminary result.

Lemma 5 *Let G be an r-graph of order n and size at least $\binom{n}{r} - n + m$, where $n > m > r \geq 2$. Then every $E \in E(G)$ is contained in a complete subgraph of order m.*

Proof. Given $E \in E(G)$, remove from each missing edge one (arbitrary) vertex not belonging to E. We are left with at least m vertices spanning a complete subgraph which contains E. $\square$

Theorem 6 *Let $l = \binom{m}{r} - k$, $c = m - k - r$, and $\mathcal{H} = \mathcal{H}_r(m,l)$. If $m > r + k$, then*

$$\text{w-sat}(n, \mathcal{H}) = c\binom{n}{r-1} + O(n^{r-2}). \tag{11}$$

Furthermore, if $r = 2$, then we have a g'-proof that

$$\text{w-sat}(n, \mathcal{H}_2(m,l)) = c(n-m) + l - 1, \quad n \geq m. \tag{12}$$

Proof. The r-graph G on $[n]$ consisting of all edges intersecting $[m-r+1]$ in at least 2 vertices plus all edges intersecting $[c]$ is weakly $\mathcal{H}$-saturated: we can add first all missing edges intersecting $[m-r+1]$ and then the remaining ones in any order. Computing $e(G)$ we obtain the upper bounds in (11) and (12).

On the other hand, in any $F \in \mathcal{H}$, any edge lies within a K^r_{m-k}-subgraph by Lemma 5. But by Theorem 4, K^r_{m-k} is a cycle in $\mathcal{G}_P$, the gross matroid of $P = P(c, n-c; 1, r-1)$, so each $F \in \mathcal{H}$ is a $\mathcal{G}_P$-cycle. By (5), $\text{w-sat}(n, \mathcal{H}) \geq R_{\mathcal{G}_P}([n]^{(r)}) = e(P)$, which g-proves the required lower bound in (11).

Finally, let us g′-prove the lower bound in (12) for $r = 2$. Let $F \in \mathcal{H}$. As F has m vertices,

$$R_{\mathcal{G}_P}(F) \leq R_{\mathcal{G}_P}(K^2_m) \leq e(P(c, m-c; 1, 1)) = cm - \binom{c+1}{2}.$$

(The second inequality is true because $P(c, m-c; 1, 1) \in \text{w-SAT}(m, K^2_{m-k})$ and K^2_{m-k} is a $\mathcal{G}_P$-cycle.) Therefore $D_{\mathcal{G}_P}(F) \geq l - cm + \binom{c+1}{2}$ and we obtain

$$\text{w-sat}(n, \mathcal{H}) \geq R_{\mathcal{G}_P}(K_n) + D_{\mathcal{G}_P}(\mathcal{F}) - 1 \geq c(n-m) + l - 1. \ \square$$

Remark. It is trivial that if $1 \leq l \leq \binom{m-1}{r} + 1$ then $\text{w-sat}(n, \mathcal{H}_r(m, l)) = l - 1$, $n \geq m$, which settles all remaining cases of Theorem 6 for $r = 2$.

Remark. In fact, the graph G constructed in the proof is weakly F-saturated, where $E(F)$ consists of the first l elements of $[m]^{(r)}$ in the colex order. So, Theorem 6 remains true if F is the only member of $\mathcal{F}$; this covers all possible cases for $r = 2$ except the trivial case $l = \binom{m-1}{2} + 1$.

3.3 Forests

By Lemma 2 we know that if $\text{w-sat}(n, T) = e(T) - 1$, then T admits a g-proof for n. In fact, we can show that we have G-proof if T is a forest.

Lemma 7 *Let F and H be any forests with $e(F) \leq e(H)$. Then F independent in $\mathcal{G}_H$. In particular, if $G \in \text{w-SAT}(n, F)$ and $e(G) = e(F) - 1$, then the pair (F, G) admits G-proof.*

Proof. We use induction on $l = e(H)$. It is enough to prove the claim for $e(F) = e(H)$. Assume that 1 is an endvertex incident to the edge $E = \{1, 2\}$ in both F and H. Clearly,

$$\det(M(H, F)) = \pm\alpha_{1,1}\alpha_{2,2} \det(M(H - E, F - E)) + (\alpha_{1,1}\text{-free polynomial}).$$

By induction we conclude that $\det(M(H, F)) \neq 0$, which proves the lemma. $\square$

If T contains, for example, vertices a, b, c of degrees $1, 1, 2$ respectively such that $\{a, c\}, \{c, d\}, \{b, d\} \in E(T)$, for some vertex d, then adding the edges $\{d, x\}$ and $\{x, y\}$ to T, any $x, y \notin V(T)$, we create each time a new graph isomorphic to T; this implies that $\text{w-sat}(n, T) = e(T) - 1$ with possible exceptions for some $n \leq 2m$. Generating a random tree by, for example, taking all m^{m-2} vertex-labelled trees equiprobable, one can show that almost every tree contains the above '*abc*-configuration' and therefore admits G-proof.

3.4 Complete Bipartite Graphs

Kalai [7] proved (in fact, g-proved) that w-sat$(n, K_{s,s}) = (s-1)n - \binom{s-1}{2}$ for $n \geq 2s+1$. Concerning $K_{s,t}$ for $t > s \geq 2$, nothing is known except trivial

$$\text{w-sat}(n, K_{s,s}) \leq \text{w-sat}(n, K_{s,t}) \leq (s-1)n + O(1). \tag{13}$$

We can show that $K_{s,t}$ is a $\mathcal{G}_G$-cycle, where $E(G)$ consists of elements of $[n]^{(2)}$ intersecting $A = [s-1]$ or lying within $B = [t-1]$. (This can be done by considering $h = (f_A^* \llcorner e_{[s]}) \wedge (f_B^* \llcorner e_{[s+1,s+t]})$.) Although this improves the lower bound in (13) for any $s \geq 2$ when t is sufficiently large, we do not have the matching upper bound, so we do not provide any further details here.

3.5 Cycles

It is not hard to prove directly the following result on cycles.

Lemma 8 *Let $n > m$. If m is odd, then* w-sat$(n, C_m) = n-1$ *and all extremal graphs are trees of order n and diameter at least $m-1$. If m is even, then* w-sat$(n, C_m) = n$ *and all extremal graphs are trees of order n of diameter at least $m-1$ plus an extra edge creating an odd cycle.*

If $n = m$, then w-sat$(n, C_m) = n$ *and all extremal graphs are obtained from a Hamiltonian cycle by adding an edge creating an odd cycle and removing any other edge.* □

In fact, any odd cycle C_m admit a g-proof for $n > m$. Indeed, consider $\mathcal{G} = \mathcal{G}_{P(1,n-1;1,1)}$. Any edge of C_m is $\mathcal{G}$-dependent on the remaining ones because the path with m edges is weakly K_3^2-saturated and K_3^2 is a $\mathcal{G}$-cycle by Theorem 4. Clearly, $R_{\mathcal{G}}([n]^{(2)}) = n-1$, so the claim follows. (If restricted to $[n]^{(2)}$, $\mathcal{G}$ is the usual *cycle matroid*.)

Even cycles admit an r-proof. Indeed, let $\mathcal{M}$ be Doob's [4] *even-cycle matroid* on $[n]^{(2)}$ which can be represented by mapping $\{i,j\} \in [n]^{(2)}$ to $e_i + e_j$, for some basis $\{e_1, \ldots, e_n\}$ of a real vector space V. It is easy to show that every even cycle is an $\mathcal{M}$-circuit and $R_{\mathcal{M}}([n]^{(2)}) = n$, which implies our claim.

Probably, even cycles do not admit a g-proof. But they do, if considered in the class of bipartite graphs, because C_4 does as a complete bipartite graph and any connected bipartite graph is weakly C_4-saturated. (See Section 5 for details.)

3.6 Disjoint Edges

The case when lK_r^r, the union of l disjoint r-edges, is forbidden is rather easy.

Lemma 9 *For $n > lr$ we have* w-sat$(n, lK_r^r) = l-1$ *and $(l-1)K_r^r$ (plus isolated vertices) is the only extremal graph. Hence, by Lemma 2, we have G-proof here.* □

However, if the forbidden graph is a perfect matching, then the exact answer is known generally for $r = 2$ only.

Theorem 10 *For $n = 2l \geq 4$, $\text{w-sat}(n, lK_2^2) = n-1$ and all extremal graphs can be obtained in the following way: complete lK_2^2 to a tree T, add an edge E creating an odd cycle and remove any edge E' contained in some perfect matching of $T + E$.* $\square$

4 Cones

The definition of the cone of an r-graph was given in the introduction. In this section we prove that cones 'preserve' G/g/g'-proofs.

Lemma 11 *Suppose that every $r-1$ vertices of an r-graph F are covered by at least one edge. If F is a $\mathcal{G}_G$-cycle, for some r-graph G, then $\mathrm{cn}(F)$ is a $\mathcal{G}_{\mathrm{cn}(G)}$-cycle.*

Proof. Suppose first that $v(G) \geq v(F)$. Let $G' = \mathrm{cn}(G)$, $V(G) = [n-1]$ and $V(G') = [n]$. Identify the vertices of G' with the basis $\{e_1, \ldots, e_n\}$ of a vector space V'. Let Z' be the subspace of $\bigwedge^r V'$ and let $\mathcal{G}_{G'}$ be the gross matroid on $[n]^{(r)}$ corresponding to G'.

We may assume that $F' = \mathrm{cn}(F)$ is embedded into $[n]$ so that $V(F') \setminus V(F) = \{n\}$. We have to show that $E(F')$ is a cycle in $\mathcal{G}_{G'}$, that is, we have to find $h' \in Z'$ such that $\mathrm{supp}(h') = E(F')$. Define $g_n^* = f_n^*$ and

$$g_i^* = f_i^* - \frac{\alpha_{in}}{\alpha_{nn}} f_n^*, \qquad i = 1, \ldots, n-1. \tag{14}$$

Recall that $\mathbf{f}^*$ is a generic basis of $(V')^*$ and $\alpha_{ij} = f_i^*(e_j)$, so $g_i^*(e_n) = 0$, $i \in [n-1]$, and this is the main point of our definition. The matrix $N = (g_i^*(e_j))_{i,j \in [n-1]}$ is clearly a generic matrix for a generic choice of the α's.

As F is a $\mathcal{G}_G$-cycle, the system of linear equations

$$g_D^* \llcorner \left(\sum_{E \in E(F)} c_E e_E \right) = 0, \qquad D \in E(G), \tag{15}$$

with respect to the undeterminants $(c_E)_{E \in E(F)}$, has a solution with all c's being non-zero for generic $\mathbf{g}$ (which is the case for generic $\mathbf{f}$). Apply elementary matrix transforms to write the system (15) in a diagonal form. For the free variables choose $\beta_1, \ldots, \beta_k$ which (together with the α's) are algebraically independent over the rationals and compute the other variables each being a rational function of the α's and β's.

Let $h = \sum_{E \in E(F)} c_E e_E$ and $h' = f_n^* \llcorner (h \wedge e_n)$. To complete the theorem it is enough to show that $h' \in Z'$ and $\mathrm{supp}(h') = E(F')$.

Let $D \in E(G')$. We want to show that $f_D^* \llcorner h' = 0$. This is clearly the case if $D \ni n$. If $D \not\ni n$, then

$$\langle f_D^*, h' \rangle = \langle \textstyle\bigwedge_{i \in D} (g_i^* + \frac{\alpha_{in}}{\alpha_{nn}} f_n^*), f_n^* \llcorner (h \wedge e_n) \rangle = \langle g_D^* \wedge f_n^*, h \wedge e_n \rangle = f_n^*(e_n) \langle g_D^*, h \rangle = 0,$$

where we used (14), the relations $g_i^*(e_n) = 0$, $i \in [n-1]$, and the definition of h. Therefore $h' \in Z'$.

Clearly, $\mathrm{supp}(h') \subset E(F')$. On the other hand, take any $E \in E(F')$. If $E \in E(F)$ then

$$\langle e_E^*, h' \rangle = \langle e_E^* \wedge f_n^*, h \wedge e_n \rangle = \langle e_E^*, h \rangle \cdot \langle f_n^*, e_n \rangle = c_E f_n^*(e_n) \neq 0,$$

because $n \notin E$. If $E \ni n$ then let $D_1, \ldots, D_l$ be the edges of F containing $E' = E \setminus \{n\}$. By our assumption, $l > 0$. Let $D_i \setminus E = \{d_i\}$. Then

$$\begin{aligned} P_E &= \langle e_E^*, h' \rangle = \langle e_{E'}^* \wedge e_n^* \wedge f_n^*, h \wedge e_n \rangle = -\langle e_{E'}^* \wedge f_n^*, h \rangle \\ &= -\Big\langle e_{E'}^* \wedge f_n^*, \sum_{E \in E(F)} c_E e_E \Big\rangle = \sum_{i=1}^{l} \pm c_{D_i} \langle f_n^*, e_{d_i} \rangle = \sum_{i=1}^{l} \pm c_{D_i} \alpha_{n,d_i}. \end{aligned}$$

(The third equality is true as $\mathrm{supp}(h) = E(F) \subset [n-1]^{(r)}$.)

As every c_{D_i} is a rational function in the α's and β's so is P_E. To show that $P_E \neq 0$ for a generic $\mathbf{f}$, it is enough to demonstrate an example of $\mathbf{f}$ when $P_E \neq 0$. Let $\alpha_{in} = 0$, $i \in [n-1]$. Then system (15) reduces to

$$f_D^* \llcorner \left(\sum_{E \in E(F)} c_E e_E \right) = 0, \qquad D \in E(G). \tag{16}$$

By the algebraic independence of $(f_i^*(e_j))_{i,j \in [n-1]}$, if we perform the diagonalization for (16) in the same order as for (15), we will obtain the same set of free variables. Therefore, $(c_E)_{E \in E(F)}$ provides every solution for (16) when the β's range over the reals. Thus each c_E is non-zero (as F is a $\mathcal{G}_G$-cycle) and it can depend only on $f_i^*(e_j) = \alpha_{ij}$, $i, j \in [n-1]$, and the β's. Now it is obvious that $P_E = \sum_{i=1}^{l} c_{D_i} \alpha_{n,d_i}$ cannot be identically zero. This proves the lemma if $v(G) \geq v(F)$.

Otherwise, we can add $v(F) - v(G)$ isolated vertices to G to obtain H. By above, $\mathrm{cn}(F)$ is a cycle in $\mathcal{G}_{\mathrm{cn}(H)}$, that is, each edge of $\mathrm{cn}(F)$ is dependent on the other edges. The latter claim is certainly true in $\mathcal{G}_{\mathrm{cn}(G)}$ which has more dependences than $\mathcal{G}_{\mathrm{cn}(H)}$ as $\mathrm{cn}(G) \subset \mathrm{cn}(H)$. $\square$

Lemma 12 *If an r-graph F is independent in $\mathcal{G}_G$ and $v(F) \leq v(G)$, then $\mathrm{cn}(F)$ is independent in $\mathcal{G}_{\mathrm{cn}(G)}$.*

Proof. We assume the same conventions as those appearing in the proof of Lemma 11 before (15). It is enough to prove our claim in the case $e(G) = e(F)$: if $e(G) > e(F)$ we can remove a G-edge with F being still $\mathcal{G}_G$-independent.

Let us show that the rank of $M'(G', F')$ is $e(F')$, where $M'(D, E) = \langle g_D^*, e_E \rangle$, $D \in E(G')$, $E \in E(F')$, which implies the lemma.

By our assumption, the square submatrix $M'(G, F) \subset M'(G', F')$ is non-singular because the matrix N is generic. As $g_i^*(e_n) = 0$ for $i \in [n-1]$, we conclude that all entries of the submatrix $M'(G, F'')$ are zeros, where $E(F'') = E(F') \setminus E(F)$. Therefore, to prove the claim we have to show that the submatrix $M'(G'', F'')$ has the maximal possible rank $\binom{v(F)}{r-1}$, where $E(G'') = E(G') \setminus E(G)$.

For any $D' = D \cup \{n\} \in E(G'')$, $E' = E \cup \{n\} \in E(F'')$, we have

$$\langle g^*_{D'}, e_{E'} \rangle = g^*_n(e_n) \cdot \langle g^*_D, e_E \rangle,$$

because $g^*_i(e_n) = 0$, $i \in [n-1]$. (As n is the last element in D' and E', we do not have ± 1 in the formula.) Now,

$$M'(G'', F'') = g^*_n(e_n) \cdot M'\big([n-1]^{(r-1)}, V(F)^{(r-1)}\big)$$

has rank $\binom{v(F)}{r-1}$ because N is generic. $\square$

Remark. Is is not hard to show that if F is not independent in $\mathcal{G}_G$, then $\mathrm{cn}(F)$ is not independent in $\mathcal{G}_{\mathrm{cn}(G)}$ for any r-graphs F and G. But we do not need this.

Lemma 13 *If $G \in$ w-SAT$(n-1, \mathcal{F})$, then $\mathrm{cn}(G) \in$ w-SAT$(n, \mathrm{cn}(\mathcal{F}))$. In particular,*

$$\text{w-sat}(n, \mathrm{cn}(\mathcal{F})) \le \text{w-sat}(n-1, \mathcal{F}) + \binom{n-1}{r-1}.$$

Proof. Let $E_1, \ldots, E_m$ be an $\mathcal{F}$-proper ordering of $E(\overline{G})$. To show that $G' = \mathrm{cn}(G)$ is weakly $\mathrm{cn}(\mathcal{F})$-saturated, add these edges in the same order to G'. (Note that $E(\overline{G'}) = E(\overline{G})$.) Every E_i creates an F-subgraph in G, $F \in \mathcal{F}$, which, together with the extra vertex, creates a copy of $\mathrm{cn}(F)$ in G', so $G' \in$ w-SAT$(n, \mathrm{cn}(\mathcal{F}))$. $\square$

Theorem 14 *Let $\mathcal{F}$ be a family of r-graphs such that in each $F \in \mathcal{F}$ every $r-1$ vertices are covered by at least one edge.*

If $(\mathcal{F}, G)$ admits G-proof, then so does the pair $(\mathrm{cn}(\mathcal{F}), \mathrm{cn}(G))$.

If we can g-prove w-sat$(n-1, \mathcal{F}) \ge l$, then we can g-prove

$$\text{w-sat}(n, \mathrm{cn}(\mathcal{F})) \ge l + \binom{n-1}{r-1}. \tag{17}$$

In particular, if $\mathcal{F}$ admits a g-proof for $n-1$, then $\mathrm{cn}(\mathcal{F})$ admits a g-proof for n. The analogous claim is true for the g$'$-technique.

Proof. The claim about G-proof follows from Lemmas 13 and 11.

Next, consider the g-technique. Take any G such that each $F \in \mathcal{F}$ is a $\mathcal{G}_G$-cycle and $R_{\mathcal{G}_G}(K^r_{n-1}) \ge l$. Adding extra vertices to G, we may assume $v(G) \ge n-1$. By Lemma 11, each graph in $\mathrm{cn}(\mathcal{F})$ is a cycle in $\mathcal{G}_{\mathrm{cn}(G)}$. By Lemma 12, $R_{\mathcal{G}_{\mathrm{cn}(G)}}(K^r_n) \ge l + \binom{n-1}{r-1}$, that is, we can g-prove (17), as required.

In the g$'$-case, choose G such that each $F \in \mathcal{F}$ is a $\mathcal{G}_G$-cycle and

$$R_{\mathcal{G}_G}(K^r_{n-1}) + D_{\mathcal{G}_G}(\mathcal{F}) - 1 \ge l.$$

Now we proceed in the same way as in the g-case, except we have to show additionally that for any $F \in \mathcal{F}$ we have $D_{\mathcal{G}_G}(F) \le D_{\mathcal{G}_{\mathrm{cn}(G)}}(\mathrm{cn}(F))$.

Note that if we have F-edges $E_1, \ldots, E_d$ whose removal does not decrease the $\mathcal{G}_G$-rank of $E(F)$ then the system of equations (15) has a solution in which $c_{E_1}, \ldots, c_{E_d}$

can be chosen to be the free variables $\beta_1, \ldots, \beta_d$. Following the proof of Lemma 11 (note that F is a $\mathcal{G}_G$-cycle), one can let (c_E) be such a solution of (15) and observe that

$$\langle e^*_{E_i}, h' \rangle = \langle e^*_{E_i} \wedge f^*_n, h \wedge e_n \rangle = \langle e^*_{E_i}, h \rangle \cdot \langle f^*_n, e_n \rangle = \beta_i \alpha_{nn}, \quad i \in [d],$$

since $E_i \subset [n-1]$. This means that, choosing generic β's, we can obtain $h' \in Z'$ whose support is $E(\mathrm{cn}(F))$ with $e^*_{E_i}(h')$ being generic, which is precisely to say that $E_1, \ldots, E_d$ are $\mathcal{G}_{\mathrm{cn}(G)}$-dependent on the other edges of $\mathrm{cn}(F)$. Hence, $D_{\mathcal{G}_{\mathrm{cn}(G)}}(\mathrm{cn}(F)) \geq d$ and the claim follows. □

Remark. We cannot generally discard the covering condition in Lemma 11 or Theorem 14. (But note that we do not have any covering condition on G.) Consider, for example, $r = 2$ when the condition rules out isolated vertices. Let F be a triangle plus an isolated vertex and let G be a star $K_{1,n-2}$, $n \geq 5$. Then (F, G) admits a G-proof (see Subsection 3.1). But it is easy to see that $\text{w-sat}(n, \mathrm{cn}(F)) = 6 < e(\mathrm{cn}(G)) = 2n - 3$, and so $\mathrm{cn}(F)$ cannot be a $\mathcal{G}_{\mathrm{cn}(G)}$-cycle.

5 Joins

Here we indicate how to extend the idea of G/g/etc.-proof to layered graphs and prove that *joins* 'preserve' G/g/r-proofs.

Let $t \in \mathbb{N}$ be fixed. A *layered set* $\mathbf{X}$ of *signature* $\mathbf{n} = (n_1, \ldots, n_t)$ is a sequence of t disjoint sets, $\mathbf{X} = (X_1, \ldots, X_t)$ such that $|X_i| = n_i$, $i \in [t]$. The components of $\mathbf{X}$ are called *layers.* Given $\mathbf{r} = (r_1, \ldots, r_t)$, a *layered* $\mathbf{r}$*-graph* $\mathbf{G}$ is a pair $(V(\mathbf{G}), E(\mathbf{G}))$ where $V(\mathbf{G})$ is a layered set and $E(\mathbf{G}) \subset V(\mathbf{G})^{(\mathbf{r})}$. In other words every $\mathbf{r}$-graph $\mathbf{G}$ is an r-graph (usually, given $\mathbf{r}$, we denote $r = \sum_{i \in [t]} r_i$, etc.) which comes with a fixed partition of the vertex set into t layers such that every edge intersects the ith layer in exactly r_i vertices. For example, bipartite graphs are $(1,1)$-graphs; for $t = 1$ we obtain the usual notion of an r-graph.

In the obvious way we define (within the class of $\mathbf{r}$-graphs) the notions of the complement, a subgraph, etc. (All morphisms, etc., between $\mathbf{r}$-graphs respect the layers.) For example, given an $\mathbf{r}$-graph $\mathbf{F}$, $\text{w-SAT}(\mathbf{n}, \mathbf{F})$ consists of all $\mathbf{r}$-graphs $\mathbf{G}$ on an $\mathbf{n}$-set such that we can add consecutively all missing $\mathbf{r}$-edges to $\mathbf{G}$ creating every time an $\mathbf{F}$-subgraph.

It is clear how to extend the notion of an m/r-proof to layered graphs. It is possible also to introduce the gross matroid of an $\mathbf{r}$-graph $\mathbf{G}$ defined on an $\mathbf{n}$-set $\mathbf{X}$. Indeed, identify each X_i with a basis $\mathbf{e}_i = (e_{i,j})_{j \in [n_i]}$ of an n_i-dimensional vector space V_i and consider $\bigwedge \mathbf{V} = \bigotimes_{i \in [t]} \bigwedge V_i$. Let $\bigwedge^{\mathbf{r}} \mathbf{V}$ be the linear subspace of $\bigwedge \mathbf{V}$ spanned by the elements

$$h = h_1 \otimes \ldots \otimes h_t, \qquad h_i \in \bigwedge^{r_i} V_i, \quad i \in [t].$$

Let $\mathbf{f}_i = (f_{i,j})_{j \in [n_i]}$ be another basis of V_i lying in generic position with respect to $\mathbf{e}_i$, $i \in [t]$. In the obvious way we define supports, etc. For any $\mathbf{r}$-subset $\mathbf{E} \subset \mathbf{X}$ let

$$\mathbf{f}_{\mathbf{E}} = \bigotimes_{i \in [t]} f_{i,E_i} \quad \text{and} \quad \mathbf{e}_{\mathbf{E}} = \bigotimes_{i \in [t]} e_{i,E_i}.$$

Let the linear subspace $\mathbf{Z} \subset \bigwedge^{\mathbf{r}} \mathbf{V}$ corresponding to $\mathbf{G}$ be spanned by the elements $\{\mathbf{f}_{\mathbf{E}} : \mathbf{E} \in E(\overline{\mathbf{G}})\}$ and let $\mathbf{r}$-sets $\mathbf{E}_1, \ldots, \mathbf{E}_k$ be independent if no linear combination of $\mathbf{e}_{\mathbf{E}_1}, \ldots, \mathbf{e}_{\mathbf{E}_k}$ (except 0) belongs to $\mathbf{Z}$. The required matroid $\mathcal{G}_{\mathbf{G}}$ of rank $\mathrm{codim}(\mathbf{Z}) = e(\mathbf{G})$ is built.

Given t (usual) r_i-graphs F_i, $i \in [t]$, with disjoint vertex sets, their *join* (or *tensor product*) $\mathbf{F} = F_1 \otimes \ldots \otimes F_t$ is the layered $\mathbf{r}$-graph on the layered set $V(\mathbf{F}) = (V(F_1), \ldots, V(F_t))$ such that an $\mathbf{r}$-subset $\mathbf{E} = (E_1, \ldots, E_t)$ is an edge of $\mathbf{F}$ if and only if $E_i \in E(F_i)$ for every $i \in [t]$. Thus $e(\mathbf{F}) = \prod_{i \in [t]} e(F_i)$.

Suppose that we are given t families $\mathcal{F}_i$ of r_i-graphs, $i \in [t]$. Define their *join* by

$$\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_t = \{F_1 \otimes \ldots \otimes F_t : F_i \in \mathcal{F}_i,\ i \in [t]\}.$$

Let these conventions apply to the remainder of this Section. We need the following two simple lemmas.

Lemma 15 *If $G_i \in \text{w-SAT}(n_i, \mathcal{F}_i)$, $i \in [t]$, then $\mathbf{G} \in \text{w-SAT}(\mathbf{n}, \mathcal{F})$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \ldots \otimes \overline{G_t}$. In particular,*

$$\text{w-sat}(\mathbf{n}, \mathcal{F}) \le \prod_{i \in [t]} \binom{n_i}{r_i} - \prod_{i \in [t]} \left(\binom{n_i}{r_i} - \text{w-sat}(n, \mathcal{F}_i) \right). \ \Box$$

Lemma 16 *If F_i is a cycle in $\mathcal{G}_{G_i}$, $i \in [t]$, then $\mathbf{F} = F_1 \otimes \ldots \otimes F_t$ is a cycle in $\mathcal{G}_{\mathbf{G}}$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \ldots \otimes \overline{G_t}$.*

Proof. By the assumption, there is $h_i \in Z_{G_i} \subset \bigwedge^{r_i} V_i$ such that $\mathrm{supp}_{\mathbf{e}_i}(h_i) = E(F_i)$, $i \in [t]$. Consider

$$h = h_1 \otimes \ldots \otimes h_t \in \textstyle\bigwedge^{\mathbf{r}} \mathbf{V}.$$

Obviously, $\mathrm{supp}_{\mathbf{e}}(h) = E(\mathbf{F})$ and $\mathrm{supp}_{\mathbf{f}}(h) \subset E(\overline{\mathbf{G}})$. Therefore, $h \in Z_{\mathbf{G}}$ and every edge in $\mathbf{F}$ is dependent on the rest, as required. $\Box$

Theorem 17 *Suppose that, for every $i \in [t]$, the pair $(\mathcal{F}_i,\ G_i)$ admit a G-proof. Then so does the pair $(\mathcal{F}, \mathbf{G})$, where $\overline{\mathbf{G}} = \overline{G_1} \otimes \ldots \otimes \overline{G_t}$.*

Suppose that, for each $i \in [t]$, we can g-prove that $\text{w-sat}(n_i, \mathcal{F}_i) \ge l_i$. Then we can g-prove that

$$\text{w-sat}(\mathbf{n}, \mathcal{F}) \ge \prod_{i \in [t]} \binom{n_i}{r_i} - \prod_{i \in [t]} \left(\binom{n_i}{r_i} - l_i \right). \tag{18}$$

In particular, by Lemma 15, if each $\mathcal{F}_i$ admits a g-proof for n_i, then $\mathcal{F}$ admits a g-proof for $\mathbf{n}$. The analogous statement is true for the r-technique.

Proof. The claim about G-proof follows from Lemmas 15 and 16.

Now, consider the g-case. For $i \in [t]$, choose G_i such that each graph in $\mathcal{F}_i$ is a $\mathcal{G}_{G_i}$-cycle and the $\mathcal{G}_{G_i}$-rank of $K_{n_i}^{r_i}$ is at least l_i; let $H_i \subset K_{n_i}^{r_i}$ be a $\mathcal{G}_{G_i}$-independent subgraph of size l_i and order n_i. Let

$$\begin{aligned} \overline{\mathbf{G}} &= \overline{G_1} \otimes \ldots \otimes \overline{G_t}, \\ \overline{\mathbf{H}} &= \overline{H_1} \otimes \ldots \otimes \overline{H_t}. \end{aligned}$$

By Lemma 16, each $F_1 \otimes \ldots \otimes F_k \in \mathcal{F}$ is a $\mathcal{G}_{\mathbf{G}}$-cycle.

Let us show that $\mathbf{H}$ is independent in $\mathcal{G}_{\mathbf{G}}$. As each H_i is $\mathcal{G}_{G_i}$-independent, we can find a linear map $p_i : \bigwedge^{r_i} V_i \to Z_{G_i}$ which is the identity map on Z_{G_i} while $p_i(e_E) = 0$ if $E \in E(H_i)$, $i \in [t]$. Define

$$\mathbf{p} = p_1 \otimes \ldots \otimes p_t : \bigwedge^{\mathbf{r}} \mathbf{V} \to Z_{G_1} \otimes \ldots \otimes Z_{G_t},$$

that is $\mathbf{p}(u_1 \otimes \ldots \otimes u_t) = p_1(u_1) \otimes \ldots \otimes p_t(u_t)$. Now, $\mathbf{p}$ is the identity map on $Z_{G_1} \otimes \ldots \otimes Z_{G_t} = Z_{\mathbf{G}}$, while $\mathbf{p}$ is zero on $e_{\mathbf{E}}$ for each $\mathbf{E} = E_1 \cup \ldots \cup E_t \in E(\mathbf{H})$. Hence, no non-zero linear combination of $e_{\mathbf{E}}$, $\mathbf{E} \in E(\mathbf{H})$ can lie in $Z_{\mathbf{G}}$, that is, $\mathbf{H}$ is independent in $\mathcal{G}_{\mathbf{G}}$. The size of $\mathbf{H}$ equals the right-hand side of (18), as required.

In the r-case, our task is to construct a matroid $\mathcal{M}$ on the set of $\mathbf{r}$-subsets of $\mathbf{X}$ such that every graph in $\mathcal{F}$ is an $\mathcal{M}$-cycle, should we be given appropriate matroids $\mathcal{M}_i$ on $Y_i = X_i^{(r_i)}$, $i \in [t]$.

Let $k_i : Y_i \to V_i$, for some vector space V_i, be a representation of the matroid $\mathcal{M}_i$, $i \in [t]$. Identify Y_i with a basis of some vector space W_i via $g_i : Y_i \hookrightarrow W_i$. Let $h_i : W_i \to V_i$ be the linear map extending k_i. Denote $Z_i = \ker(h_i) \subset W_i$. Clearly, $\mathrm{codim} Z_i = R_{\mathcal{M}_i}(Y_i) = e(G_i) \geq l_i$, where G_i is a base of $\mathcal{M}_i$.

Let $\overline{\mathbf{G}} = \overline{G_1} \otimes \ldots \otimes \overline{G_t}$. Identify the $\mathbf{r}$-subsets of $V(\mathbf{G})$ with a basis of $\mathbf{W} = \bigotimes_{i \in [t]} W_i$ by mapping $\mathbf{E} = (E_1, \ldots, E_t)$ into $g(\mathbf{E}) = \bigotimes_{i \in [t]} g_i(E_i)$. Let $\mathbf{Z} = \bigotimes_{i \in [t]} Z_i \subset \mathbf{W}$ and $p : \mathbf{W} \to \mathbf{W}/\mathbf{Z}$ be the projection.

Let $\mathcal{M}$ be the matroid represented by $p \circ g : V(\mathbf{G})^{(\mathbf{r})} \to \mathbf{W}/\mathbf{Z}$. Let us show that $\mathcal{M}$ r-proves (18).

As $g(V(\mathbf{G})^{(\mathbf{r})})$ is a basis for $\mathbf{W}$, we conclude that the rank of $\mathcal{M}$ is

$$\dim \mathbf{W} - \dim \mathbf{Z} = \prod_{i \in [t]} \binom{n_i}{r_i} - \prod_{i \in [t]} e(\overline{G_i}),$$

which is at least the right-hand side of (18).

Thus, all we have to do is to check that any $\mathbf{F} = F_1 \otimes \ldots \otimes F_t \in \mathcal{F}$ is an $\mathcal{M}$-cycle. Fix an edge $\mathbf{E} = (E_1, \ldots, E_t) \in E(\mathbf{F})$. As F_i is an $\mathcal{M}_i$-cycle, we conclude that there are $c_{i,E} \in \mathbb{R}$, $E \in E(F_i) \setminus \{E_i\}$, and $z_i \in Z_i$ such that

$$g_i(E_i) = z_i + \sum_{D \in E(F_i) \setminus \{E_i\}} c_{i,D}\, g_i(D), \qquad i \in [t]. \tag{19}$$

If we take the tensor product of (19) over $i \in [t]$, we obtain on the left-hand side the element $g(\mathbf{E})$ while on the right-hand side we will have $z_1 \otimes \ldots \otimes z_t \in \mathbf{Z}$ plus some other tensor products. Next, in the remaining tensor products replace each z_i by the linear combination of $(g_i(D))_{D \in E(F_i)}$ by (19). Each term then becomes $\otimes_{i \in [t]} g_i(D_i)$ for some $D_i \in E(F_i)$, i.e., it is of the form $g(\mathbf{D})$, $\mathbf{D} = (D_1, \ldots, D_t) \in E(\mathbf{F})$ and, moreover, we never have $\mathbf{D} = \mathbf{E}$. So we have a representation of $g(\mathbf{E})$ as a linear combination of an element of $\mathbf{Z}$ and of $g(\mathbf{D})$, $\mathbf{D} \in E(\mathbf{F}) \setminus \{\mathbf{E}\}$ which is precisely the required. The theorem is proved. □

Unfortunately, there does not seem to be a natural tensor product operation for matroids, cf. Lovász [10], so we do not know if joins preserve m-proofs.

Acknowledgements. The author is grateful to Andrew Thomason for the very helpful discussions and comments.

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