Characterization of Product Anti-Magic Graphs of Large Order

Oleg Pikhurko*
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213
Web: `http://www.math.cmu.edu/~pikhurko`

Abstract

A graph G is *product anti-magic* if one can bijectively label its edges with integers $1, \ldots, e(G)$ so that no two vertices have the same product of incident labels. This property was introduced by Figueroa-Centeno, Ichishima, and Muntaner-Batle who in particular conjectured that every connected graph with at least 4 vertices is product anti-magic.

Here, we completely describe all product anti-magic graphs of sufficiently large order, confirming the above conjecture in this case. Our proof uses probabilistic methods.

1 Introduction

We will use the standard notation of graph theory, which can be found e.g. in Bollobás' book [3].

Let G be a graph. For the purposes of this paper, the term *labeling* will always mean a bijection $\ell : E(G) \to [e(G)]$, where we denote $[n] = \{1, \ldots, n\}$. In other words, we label edges of G with integers $1, \ldots, e(G)$ so that each integer is used exactly once.

Following Figueroa-Centeno, Ichishima, and Muntaner-Batle [5], let us call the graph G *product anti-magic* if there is a labeling ℓ such that for any two distinct vertices

*Partially supported by the National Science Foundation, Grant DMS-0457512.

$x, y \in V(G)$ we have $\Pi(x) \neq \Pi(y)$, where

$$\Pi(x) = \prod_{z \in \Gamma(x)} \ell(xz)$$

denotes the product of all labels incident to x and xz is a shortcut for $\{x, z\}$. (This abbreviation should not cause any confusion, hopefully.)

The property of identifying each vertex of a graph by a unique value helps in various situations (e.g. for the problem of describing graphs in first order logic, see Bohman et al [2, 9]). It also appears on its own in many graph labeling problems (see, for example, the excellent survey by Gallian [6]).

Figueroa-Centeno et al [5] demonstrated that various special classes of graphs are product anti-magic: 2-regular graphs, wheels, complete graphs, paths of of order at least 4, and other. They also posed the following conjecture.

Conjecture 1 (Figueroa-Centeno et al [5]) *Every connected graph of order at least 4 is product anti-magic.*

Kaplan, Lev, and Roditty [8] verified Conjecture 1 for further families of graphs (dense graphs, complete bipartite graphs, and other).

Here we will completely describe all product anti-magic graphs of sufficiently large order. Let the *spider* $S_{k,l}$ be obtained by affixing k single edges and l paths of length 2 to a fixed vertex x, called the *center*. (Alternatively, subdivide some l edges of the star $K_{1,k+l}$.)

Theorem 2 *There is an n_0 such that for every $n \geq n_0$ the following characterizes all graphs of order n that are **not** product anti-magic.*

1. *Graphs that have at least one isolated edge.*
2. *Graphs that have at least two isolated vertices.*
3. *Unions of vertex-disjoint $K_{1,2}$'s.*
4. *Graphs consisting of one isolated vertex and vertex-disjoint spiders.*

Let us show now the trivial part of Theorem 2, namely that every graph G on the list (in fact, of an arbitrary order $n \geq 2$) does not admit a product anti-magic labeling ℓ.

1. If xy is an isolated edge, then $\Pi(x) = \Pi(y) = \ell(xy)$.

2. If x and y are isolated vertices, then $\Pi(x) = \Pi(y) = 1$.

3. Let xy have the label 1 and let xz be the other edge of $K_{1,2}$. Then $\Pi(x) = \Pi(z) = \ell(xz)$.

4. Let w be the isolated vertex and let xy have the label 1. If xy is a leaf, say $d(y) = 1$, then $\Pi(y) = \Pi(w) = 1$. So assume that x is the center of a spider and y has another neighbor z. Then $\Pi(y) = \Pi(z) = \ell(yz)$.

The proof of the other implication of Theorem 2 occupies Sections 2 and 3. This is done by means of a probabilistic labeling algorithm; its overview can be found at the beginning of Section 3. Our calculations (omitted) indicate that, for example, the value $n_0 = 10^{10^{20}}$ would suffice. This bound can be definitely lowered. However, it seems that this approach would not give the complete characterization, of a reasonable length, of all product anti-magic graphs. Therefore, we make no attempt to optimize the constants.

Theorem 2 implies in particular that every connected graph of order at least n_0 is product anti-magic, that is, we have verified Conjecture 1 for all but finitely many graphs.

Our proof is quite versatile. It can deal with some modifications of the problem. For example, if we wish to describe all graphs G of large order for which there is a bijection $\ell : E(G) \to \{2, \ldots, e(G) + 1\}$ with all products $\Pi(x)$, $x \in V(G)$, being distinct, then our method shows that these are precisely the graphs not in Lists 1 and 2 of Theorem 2.

There is the related conjecture of Hartsfield and Ringel [7, p. 108] that every connected graph G of order $n \geq 3$ is *sum anti-magic*, that is, admits a labeling such that for any distinct $x, y \in V(G)$ the sums of incident labels are different. Alon, Kaplan, Lev, Roditty, and Yuster [1] proved this conjecture for various classes including all graphs of minimum degree at least $C \log n$ for some constant C. Unfortunately, our method does not apply to this problem.

2 Notation and Auxiliary Lemmas

Let $\pi(m)$ denote the number of primes less or equal to m. Let p_i be the i-th prime. Let ln denote the natural logarithm. Let $a \pm c$ denote a number between $a - c$ and $a + c$.

For a graph G, let $d(x)$ denote the degree of a vertex $x \in V(G)$ and for a set of edges $F \subset E(G)$, let $V(F) = \cup_{D \in F} D$. Given a labeling ℓ of G, two vertices x, y are *distinguishable* if $\Pi(x) \neq \Pi(y)$. A vertex $x \in V(G)$ is *identifiable* if it is distinguishable from any other vertex, that is, $\Pi(x) \neq \Pi(y)$ for any $y \in V(G) \setminus \{x\}$. Thus a labeling is product anti-magic if every vertex is identifiable.

Here we list some auxiliary results that will be needed in the proof of Theorem 2. Since we are content just to prove the existence of the constant n_0 in Theorem 2, without an explicit bound on it, the following classical result from number theory will suffice for this purpose.

Lemma 3 (The Prime Number Theorem) *For every $\varepsilon > 0$ there is an $m_1 = m_1(\varepsilon)$ such that for every $m \geq m_1$ we have*

$$p_m = (1 \pm \varepsilon)\, m \ln m \quad \text{and} \quad \pi(m) = (1 \pm \varepsilon)\, \frac{m}{\ln m}. \blacksquare$$

The following lemma estimates how likely the product of two random numbers is to hit any given target t.

Lemma 4 *For every $\varepsilon > 0$ here is an $m_2 = m_2(\varepsilon)$ such that for any $m \geq m_2$ the following holds. Let $Q \subset [m]$ be any subset of size $q \geq m/4$. Let $a_1 < a_2$ be random elements of Q, all $\binom{q}{2}$ choices being equally probable. Then for any integer t, we have*

$$\Pr\{a_1 a_2 = t\} \leq \frac{(4+\varepsilon)^{\ln m/\ln\ln m}}{m^2}. \tag{1}$$

Proof. Let m be sufficiently large. If $t \geq m^2$, then the required probability is 0, so let us assume otherwise.

Clearly, the probability in question is at most $\rho/\binom{q}{2}$, where ρ is the number of divisors of t. Take the prime factorization $t = \prod_{i \in I} p_i^{\mu_i}$, where I consists of those indexes i for which $\mu_i > 0$. Then $\rho = \prod_{i \in I}(\mu_i + 1)$. Define

$$s = \frac{\ln m}{(\ln\ln m)^2}, \quad I' = \{i \in I : p_i \leq s\}, \quad I'' = I \setminus I'.$$

Note that, by Lemma 3,

$$|I'| \leq \pi(s) \leq \frac{2s}{\ln s}.$$

Moreover, for each $i \in I'$ we have $\mu_i \leq \log_2(m^2)$. Hence,

$$\prod_{i \in I'} (\mu_i + 1) \leq \left(\log_2(m^2) + 1\right)^{2s/\ln s} = \mathrm{e}^{o(\ln m/\ln\ln m)}. \tag{2}$$

Let us turn to I''. We have

$$\sum_{i \in I''} \mu_i \leq \ln t/\ln s.$$

Given the sum of positive integers μ_i, if we want to maximize the product $\prod(\mu_i+1)$, then we have to take each μ_i to be 1. (Otherwise, replace $\mu_i \geq 2$ by $\mu_i - 1$ and 1.) Hence,

$$\prod_{i \in I''} (\mu_i + 1) \leq 2^{\ln t/\ln s} \leq (4+o(1))^{\ln m/\ln\ln m}. \tag{3}$$

The estimates (2) and (3) finish the proof. ∎

Remark. The constant 4 in the bound (1) is best possible. Here is a sketch of the proof. Let m be large and $Q = [m]$. Take the smallest l such that $p_l \geq \ln m$ and let $t = \prod_{i=l}^{k} p_i$, where k is as large as possible provided $t < m^{2-1/\ln\ln m}$. One can check that

$$k - l = (2+o(1))\,\frac{\ln m}{\ln\ln m}.$$

Also, by Chernoff's bound [4], almost all sums $\sum_{i=l}^{k} b_i \ln p_i$ with $b_i \in \{0,1\}$ are within, for example, $(k-l)^{2/3}\ln\ln m$ from the mean value $\frac{1}{2}\ln t$. Hence, there are at least

$$(1+o(1))\,2^{k-l+1} = (4+o(1))^{\ln m/\ln\ln m}$$

ways to factor $t = d_1 d_2$ with $1 \leq d_1 < d_2 \leq \exp(\frac{1}{2}\ln t + (k-l)^{2/3}\ln\ln m) \leq m$.

We will also need the following easy result.

Lemma 5 *There is an m_3 such that for any $m \geq m_3$ and any subset $Q \subset [m]$ of size $q \geq m/2$ the following holds. Let a_1, a_2 be two distinct elements of Q chosen uniformly at random. Then*

$$\Pr\{a_1 a_2 \leq m\} \leq \frac{5\,\ln m}{m}.$$

Proof. Given a_1, there are at most m/a_1 choices of a_2 satisfying $a_1 a_2 \leq m$. Hence, the required probability is at most

$$\frac{1}{q}\sum_{a_1=1}^{m} \frac{m/a_1}{q-1} \leq \frac{4}{m-2}\sum_{a_1=1}^{m}\frac{1}{a_1} \leq \frac{5\ln m}{m}.$$

The lemma is proved. ∎

3 Proof of Theorem 2

Here we will prove Theorem 2. Let n be sufficiently large and let G be an arbitrary graph of order n and size m that does not appear in any forbidden list of Theorem 2. As we do not have two isolated vertices or an isolated edge,

$$m \geq \frac{2n}{3} - 1. \tag{4}$$

Let L consist of all primes between $m/2+1$ and m. By Lemma 3 we have

$$|L| \geq \frac{m}{3\ln m}. \tag{5}$$

Let us briefly outline our argument. The existence of a product anti-magic labeling will be established by means of a probabilistic algorithm which consists of three stages and, in fact, produces the required labeling with probability at least $1-c_n$, where c_n does not depend on G and approaches 0 as $n \to \infty$. In Stage 1 we (deterministically) construct disjoint sets $R, F \subset E(G)$ and assign to $R \cup F$ some labels from L. In Stage 2 we randomly extend the partial labeling ℓ to the whole of $E(G)$. There may be some pairs xy with $\Pi(x) = \Pi(y)$. We correct each such pair xy in Stage 3 by swapping some labels incident to x or y with some labels from F. Note that if some edge uv is assigned a label from L then automatically $\Pi(u)$ differs from any other $\Pi(w)$ with $w \notin \{u, v\}$ because $\Pi(w)$ cannot contain $\ell(uv)$ as a prime factor. Hence, an L-label borrowed from F is enough to repair one bad pair xy. On the other hand, we take care to secure each vertex $x \in V(F)$ by surrounding it by R-edges in Stage 1, so that x stays identifiable even if the labels of F change. This finishes our rough outline. The real proof is more complicated since we have to treat vertices of degree 1 as well as the edge e_1 with $\ell(e_1) = 1$ in a special manner.

Now, let us formally state the properties of F, R, e_1 that we will need and prove that such sets always exist. A labeling ℓ is called (R, e_1)-*proper* if $\ell(e_1) = 1$ and $\ell(R) \subset L$. Let

$$f = \lfloor n^{1/3}/4 \rfloor. \tag{6}$$

Lemma 6 *There are disjoint $R, F, \{e_1\} \subset E(G)$ such that all the following conditions hold.*

1. *$|F| = f$ and $|R| \leq 4f + 4$.*
2. *If G has an isolated vertex, then e_1 is not a leaf.*
3. *For every (R, e_1)-proper labeling ℓ, any $x \in V(F \cup R) \cup e_1$ is identifiable or has degree* 1.

Proof. We start with $R = F = \emptyset$ and will be iteratively enlarging these sets. In the proof we will have to check various alternatives. In each case it will be straightforward to see why the vertices in $V(F \cup R) \cup e_1$ satisfy Condition 3 of the lemma. We will justify a few such claims. But most of the time we will be leaving all routine verifications to the Reader who should not have any problem filling up the gaps.

First we define e_1. If G contains a cycle C on vertices $x_1, \ldots, x_s$, then for $s \leq 5$ we let $e_1 = x_1x_2$ and put all other edges of C into R and for $s \geq 6$ we let $e_1 = x_3x_4$ and put $x_1x_2, x_2x_3, x_4x_5, x_5x_6$ into R. For example, to argue that the vertex x_4 is identifiable for $s \geq 6$ observe that $\Pi(x_4) \neq \Pi(x_5)$ because the latter is divisible by the prime $\ell(x_5x_6) \in L$; also, any $\Pi(x)$ for $x \notin \{x_4, x_5\}$ is not divisible by $\ell(x_4x_5) \in L$ which, however, divides $\Pi(x_4)$. So suppose that G is a forest. If G has no isolated vertex then, since G is not in Lists 1 and 3 of Theorem 2, we can find x, y, x, w such that $d(x) = 1$ and either $xy, yz, zw \in E(G)$ or $xy, yz, yw \in E(G)$. In both cases we let $e_1 = xy$ and put the other two edges into R. Finally, if the forest G has an isolated vertex, then a routine analysis shows that there are $wx, xy, yz \in E(G)$ and vertices $u, v \notin \{w, x, y, z\}$ such that u sends an edge to wx and v sends an edge to yz. In this case we let $e_1 = xy$ and put the remaining 4 edges into R.

Now, let us describe how to construct F and R. We start with $F = \emptyset$ and R being the set of at most 4 edges used for 'padding' e_1.

Suppose first that the maximal degree $\Delta(G) \geq f + 3$. Take a vertex x of maximum degree. Make sure that some two edges incident to x belong to R; then x is automatically identifiable. We can do always this in such a way that at most 3 edges at x have been assigned so far (the two R-edges and, perhaps, e_1). Put some f of the remaining edges incident to x into F.

Take any $xy \in F$. We have to put some edges into R to ensure the claimed properties for y. If $d(y) = 1$ or $y \in e_1$, then there is nothing to do. If $d(y) \geq 3$, make sure that at least some two edges incident to y are in R; this can be done by adding at most two new edges to R. Suppose that $d(y) = 2$. Let $z \neq x$ be the other neighbor of y. If $d(z) = 1$, put yz into R which takes care of y: for example,

$$\Pi(y) = \ell(yz)\ell(xy) > \ell(yz) = \Pi(z)$$

for any ℓ with $\ell(e_1) = 1$. If z has a neighbor $w \notin \{x, y\}$, then we put both yz and zw into R. Finally, if y and x are the only neighbors of z, then we put yz into R and everything is fine: for example,

$$\Pi(y) = \ell(xy)\ell(yz) \neq \ell(xz)\ell(yz) = \Pi(z),$$

because edge labels are distinct. Thus x alone supplies us with the required F and R (note that $|R| \leq 2|F| + 6 \leq 4f + 4$).

Therefore let us assume that $\Delta(G) \leq f + 2$. We proceed iteratively, enlarging the sets F and R as we go along. It will always be the case that $|R| \leq 4|F| + 4$.

Here is the description of the iteration step. As long as $|F| < f$, take any edge xy at distance at least 3 from any edge previously selected into $F \cup R \cup \{e_1\}$, where the distance between two edges is the usual distance in the line graph of G. Such an edge xy exists because at most

$$(|F \cup R| + 1) \times 2(f+1)(f+2) < e(G)$$

edges are excluded by the assumption on maximum degree. If xy lies in a triangle xyz, then we can put xy into F and xz, yz into R, so suppose otherwise.

If xy lies on a path $wxyz$, then we do the following. Put xy in F and wx, yz in R. If w has a neighbor $u \neq x$ (including the case $u = y$), then we choose one such u and put wu into R. Likewise, if z has a neighbor $v \neq y$, then we put zv into R.

It remains to consider the case when e.g. $d(x) = 1$. If $d(y) \geq 3$, put some two edges incident to y into R. Otherwise we have $d(y) = 2$ (because G does not have an isolated edge). If the other neighbor z of y has a neighbor $w \neq y$, put xy into F and yz, zw into R; otherwise we have an isolated $K_{1,2}$ so put xy into F and yz into R.

In each of the above cases, any edges we assign are at distance at most 2 from xy so no collision with the previous assignments can arise. Also, we add at most four edges to R per one edge of F.

This completes the proof of the lemma. ∎

Lemma 6 takes care of Stage 1: take any F, R, and e_1 given by the lemma and arbitrarily assign some labels from L to $F \cup R$ (and the label 1 to e_1).

In Stage 2 we randomly extend this partial labeling to the whole of $E(G)$, all possible extensions being equally likely. Let us denote the resulting random labeling by ℓ.

Call a vertex $x \in V(G)$ *thin* if $x \notin V(F \cup R) \cup e_1$, $d(x) \geq 2$, and $\Pi(x) \leq m$. Let $\mathcal{T}$ consist of all thin vertices. By Lemma 5 and (4) the expected value of $|\mathcal{T}|$ is at most

$$n \times \frac{5 \ln m}{m} \leq 8 \ln m.$$

(Note that the number of the random labels assigned in Stage 2 is at least $m - 5f - 5 \geq m/2$ by (4) and (6).)

Call a pair of distinct vertices xy *bad* if $x, y \notin V(F \cup R) \cup e_1$, and $\Pi(x) = \Pi(y)$. Otherwise, we call xy *good*. Let $\mathcal{B}$ consist of all bad pairs.

Let us estimate the probability that two given vertices $x, y \notin V(F \cup R) \cup e_1$ form a bad pair. Let $d'(z)$ be the number of neighbors of z outside $\{x, y\}$. Assume that $d'(x) \geq d'(y)$. We cannot have $d'(x) = d'(y) = 0$ because this would give either two isolated vertices or an isolated edge. If $d'(x) = d'(y) = 1$, then $\Pi(x) \neq \Pi(y)$ because all edge labels are distinct. If $d'(x) = 1$ and $d'(y) = 0$, say $xz \in E(G)$, then x and y are distinguishable because we have $\ell(xz) \neq 1$. (Note that $x \notin e_1$.) It remains to consider the case when $d'(x) \geq 2$. Pick some two edges $xu, xv \in E(G)$ with $\{u,v\} \not\ni y$. Let us view Stage 2, the random extension, as a two-step process. At the first step we assign random labels to all edges incident to $\{x,y\}$ except the edges xv, xu. Conditioned on this, $\ell(xu)$ and $\ell(xv)$ are two random distinct elements of the set Q of the remaining labels. There is at most one possible value of the product of these two labels that makes $\Pi(x) = \Pi(y)$.

If $\Delta(G) \leq n/5$, then Condition 1 of Lemma 6 and (4) imply that

$$|Q| \geq e(G) - |F \cup R| - 1 - 2\Delta(G) \geq m - 5f - 5 - 2n/5 \geq m/4.$$

If $\Delta(G) > n/5$, then the proof of Lemma 6 shows that at least one vertex z of maximum degree belongs to $V(R)$ and that at least $\Delta(G) - f - 4$ edges incident to z were not assigned labels in Stage 1. Since $x, y \notin V(R)$, we have $z \notin \{x,y\}$ and thus

$$|Q| \geq \max\Big(m - 5f - 5 - 2\Delta(G),\, \Delta(G) - f - 6\Big) \geq m/4.$$

We see that in all cases where the equality $\Pi(x) = \Pi(y)$ is not ruled out automatically, we have $|Q| \geq m/4$. Lemma 4 applies to the random labels $\ell(xu)$ and $\ell(xv)$ from Q and shows that the probability of xy being bad is at most, for example, $(5^{\ln m/\ln\ln m})/m^2$. Therefore, the expectation of the number of bad pairs plus the number of thin vertices is

$$E\big(\,|\mathcal{B} \cup \mathcal{T}|\,\big) \leq \binom{n}{2} \times \frac{5^{\ln m/\ln\ln m}}{m^2} + 8\ln m \leq f.$$

We conclude that there is an extension ℓ such that

$$|\mathcal{B} \cup \mathcal{T}| \leq f. \tag{7}$$

(In fact, we have $E(\,|\mathcal{B} \cup \mathcal{T}|\,) = o(f)$, so almost every extension satisfies this property.) Fix a labeling ℓ satisfying (7).

In Stage 3 we modify this ℓ, making it product anti-magic.

First, we eliminate all thin vertices by repeating the following step as long as possible. If there is a thin vertex x, take some its neighbor y and swap the label of xy with an L-label of some edge in F. Since $d(x) \geq 2$, $\min L > m/2$, and $x \notin e_1$, the vertex x is not thin anymore. Note that this operation cannot create any new bad pairs (nor new thin vertices). Indeed, the affected vertices x and y can be distinguished from any other vertex $z \notin \{x,y\}$ because $\ell(xy)$ belongs to L now. Also, if the pair xy was good, it stays so.

Next, we iteratively eliminate all bad pairs as follows. Each bad pair xy, say with $d'(x) \geq d'(y)$, satisfies $d'(x) \geq 2$ and can be 'repaired' by swapping the label of some edge

$xz \neq xy$ with an L-label of an edge in F. As before, this does not create any new thin vertices or bad pairs.

By (7), we can eliminate all thin vertices and bad pairs. We claim that the obtained labeling ℓ is product anti-magic. It remains to check that any two vertices $x \in V(F \cup R) \cup e_1$ and $y \in V(G)$ have different products. If $d(x) \geq 2$, this follows from Lemma 6. (Note that ℓ is still (R, e_1)-proper.) Assume that $d(x) = 1$. If $d(y) = 1$, then $\Pi(x) \neq \Pi(y)$ holds since edge labels are pairwise distinct. So assume that $d(y) \geq 2$. If $y \in V(F \cup R) \cup e_1$ then y is identifiable by Lemma 6; otherwise $\Pi(y) > m \geq \Pi(x)$ because y is not thin.

This completes the proof of Theorem 2. ∎

Acknowledgments

The author thanks Benny Sudakov for helpful remarks.

References

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster, *Dense graphs are antimagic*, J. Graph Theory **47** (2004), 297–309.

[2] T. Bohman, A. Frieze, T. Łuczak, O. Pikhurko, C. Smyth, J. H. Spencer, and O. Verbitsky, *First order definability of trees and sparse random structures*, Combin. Prob. Computing **16** (2007), 375–400.

[3] B. Bollobás, *Modern graph theory.*, Springer-Verlag, Berlin, 1998.

[4] H. Chernoff, *A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations.*, Ann. Math. Statistics **23** (1952), 493–507.

[5] R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, *Bertrand's postulate and magical product labelings*, Bull. ICA **30** (2000), 53–65.

[6] J. A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Combin. **DS6** (2005), 148pp.

[7] H. Hartsfield and G. Ringel, *Pearls in graph theory*, Academic Press, 1990.

[8] G. Kaplan, A. Lev, and Y. Roditty, *Bertrand Postulate, the Prime Number Theorem, and product anti-magic graphs*, Talk presented at the 20th British Combinatorial Conference, 2005.

[9] J. H. Kim, O. Pikhurko, J. Spencer, and O. Verbitsky, *How complex are random graphs in first order logic?*, Random Struct. Algorithms **26** (2005), 119–145.