Odd and Even Cycles in Maker-Breaker Games

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Abstract

Let Maker and Breaker alternately select respectively 1 and q previously unclaimed edges of K_n until all edges have been claimed. In the *even cycle game* Maker's aim is to create an even cycle. We show that if $q < \frac{n}{2} - o(n)$, then Maker has a winning strategy. This is asymptotically matched by a previous result of the authors [1] that if $q \geq \lceil n/2 \rceil - 1$ then Breaker can ensure that Maker's graph is acyclic. We also consider the *odd cycle game* and show that for $q < (1 - 1/\sqrt{2} - o(1))n$ Maker can create an odd cycle.

1 Introduction

Maker and Breaker claim respectively 1 and q edges of K_n until all edges have been claimed. For brevity, we shall refer to Maker and Breaker as *he* and *she* respectively. Also, let us assume

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that Maker colors his edges red and Breaker blue.

The more general results of the authors [1] imply that Maker has a strategy which guarantees a red cycle if and only if

$$q < \lceil n/2 \rceil - 1. \tag{1}$$

(It happens that for this particular game the critical q does not depend on who starts the game.)

In this paper we investigate the *even cycle game* (resp. *odd cycle game*) where Maker's aim is to create an even (resp. odd) cycle while Breaker tries to prevent this. It is easy to see that in each of these games (once we fix the player who starts) there is the *threshold* $q_0 = q_0(n)$ such that Maker has a winning strategy if and only if $q < q_0$. We were not able to determine the exact threshold for the even/odd cycle games, but we have proved the following lower bounds.

Theorem 1 *For every $\varepsilon > 0$ if $q < (1-1/\sqrt{2}-\varepsilon)n$ and n is big enough, then Maker can create an odd cycle.*

Theorem 2 *Let $\varepsilon > 0$ be arbitrary. For any $n > n_0(\varepsilon)$, if*

$$q < \frac{n}{2} - \left(\frac{1}{2} + \varepsilon\right) \frac{n \log \log n}{\log n},$$

then Maker can create an even cycle.

Of course, a cycle of specific parity is a more restrictive configuration than an arbitrary cycle, so the thresholds on q in our games are at most the threshold (1). We were not able to improve on the upper bound (1) for these games except a tiny improvement for the even cycle game on K_{2m+1} with Maker starting, see Theorem 4. We could not settle even the following question.

Problem 3 *In the odd cycle game, does the threshold equal $(\frac{1}{2} + o(1))\, n$?*

2 Odd Cycle Game

Let us prove Theorem 1. Assume that Breaker starts the game since this makes our task harder. Let $\varepsilon > 0$, $\alpha = 1 - 1/\sqrt{2}$, n be sufficiently large, and $q < (\alpha - \varepsilon)n$. Suppose on the contrary to the claim that Breaker always wins.

Let us describe a strategy for Maker. Maker's graph will always be a tree T with bipartition $V_1 \cup V_2$ except when Maker can immediately create an odd cycle and win the game. Let

$R = V(K_n) \setminus (V_1 \cup V_2)$. The sets V_1 and V_2 (and R) will be updated after each round. We assume that Breaker have claimed all edges inside V_1 and all edges inside V_2 after each her move (otherwise Maker wins in one move).

At the first stage of the game, Maker arbitrarily enlarges his tree until $|V_1| = \lceil \beta n \rceil$ or $|V_2| = \lceil \beta n \rceil$, where

$$\beta = \frac{1}{\sqrt{2}} - \frac{1}{2}.$$

Observe that if Breaker could prevent Maker from achieving this, then all edges between $V(T)$ and R would be selected by Breaker after her t-th round for some $t \le 2\beta n$. So

$$\begin{aligned} qt &\ge \binom{|V_1|}{2} + \binom{t-|V_1|}{2} + t(n-t) \ \ge\ \frac{t^2}{4} - \frac{t}{2} + t(n-t) \\ &\ge \left(1 - \frac{3\beta}{2}\right) nt - \frac{t}{2} \ >\ \alpha nt - \frac{t}{2}, \end{aligned}$$

which for n big enough contradicts the assumption that $q < (\alpha - \varepsilon)n$. Here we have a lot of room to spare. In fact, the optimal β is determined by (2).

Suppose that, for example, $|V_1| = \lceil \beta n \rceil$ after the first stage. In the second stage, Maker's strategy is to make V_2 as large as possible. Namely, in every move he selects an edge between V_1 and R. At some point, this strategy must stop. Let us analyze this situation.

Let $v_1 = |V_1|$ and $t = |V_1| + |V_2|$. Breaker must have claimed all edges inside V_1 and V_2 and all edges between V_1 and R. Hence, the number of Breaker's edges is

$$qt \ge \binom{v_1}{2} + \binom{t-v_1}{2} + v_1(n-t), \tag{2}$$

which for n big enough implies that

$$\alpha nt > \frac{\beta^2 n^2}{2} + \frac{(t-\beta n)^2}{2} + \beta n(n-t).$$

A solution for the above quadratic strict inequality with a real t exists if and only if the discriminant is positive, that is, if

$$n^2 \left((2\beta + \alpha)^2 - 2\beta^2 - 2\beta\right) > 0.$$

However, this is not true for $\alpha = 1 - 1/\sqrt{2}$ and $\beta = 1/\sqrt{2} - 1/2$. This is a contradiction proving Theorem 1.

3 Even Cycle Game

3.1 Proof of Theorem 2

Let Breaker start the game. Let $\varepsilon > 0$ be sufficiently small and $n > n_0(\varepsilon)$ be large.

We have to describe Maker's strategy. Define $m = \lfloor \frac{(1-\varepsilon)\log n}{\log\log n} \rfloor$ and $d = \lfloor \frac{3\log n}{\varepsilon \log\log n} \rfloor$. Maker's edges will be of two colors: green and red, that is, Maker's graph M will be the edge-disjoint union of graphs M_g and M_r. Maker tries to ensure that all of the following conditions hold.

1. The green graph is a forest of maximum degree at most d.
2. The addition of any edge from M_r to M_g creates the (unique) cycle of length at least m.

And, of course, if Maker can create an even cycle, he does so and wins the game.

Suppose that Breaker can beat this strategy of Maker. This implies that all cycles of M are odd, two cycles share at most one vertex, and the girth of M is at least m.

The double counting of the number of pairs $(e_r, e_g) \in E(M_r) \times E(M_g)$ such that e_g is on the (unique) cycle of $M_g + e_r$ shows that $(m-1)\, e(M_r) \le e(M_g)$. Thus, if t is the number of Maker's moves so far, then

$$t \le e(M_g) + \frac{e(M_g)}{m-1} \le \frac{m(n-1)}{m-1}.$$

If all edges of K_n have been colored, then $q(t+1) \ge \binom{n}{2} - t$, giving the required. So, let us analyze a position when it is Maker's turn but he cannot keep the above properties by adding either a green or red edge.

Let $H \subset [n]$ be the set of vertices of green degree d in the forest M_g. It is easy to show that

$$|H| \le \frac{n-2}{d-1}.$$

Breaker's graph must include all edges lying within $V \setminus H$ with the exception of edges of $E(M)$ (at most $t \le \frac{m(n-1)}{m-1}$ edges) and edges connecting two vertices at distance at most $m-2$ in M_g (roughly, at most $d^m n$ edges). Hence, by counting Breaker's edges, we obtain

$$q\left(\frac{m(n-1)}{m-1} + 1\right) \ge q(t+1) \ge \binom{n - \frac{n-2}{d-1}}{2} - \frac{m(n-1)}{m-1} - d^m n.$$

The theorem routinely follows.

3.2 Improving the Upper Bound (1)

For a while we believed that the threshold for creating an even cycle is exactly the same as for creating any cycle. But we can show that they differ when n is odd and Maker starts.

Theorem 4 *Let $n = 2m+1$ and let Maker start the even cycle game on K_n. If $q \geq m-1$, then Breaker has a winning strategy.*

Proof. Let $\mathcal{M}$ be the *Doob's even cycle matroid* [2] on $\binom{[n]}{2}$ (see e.g. [4, Exercise 12.2.13]). Namely, a graph $G \subseteq K_n$ is independent if each component has at most one cycle and G has no even cycles. Clearly, $\mathrm{rank}_{\mathcal{M}}(K_n) = n$. It is well-known that K_n can be decomposed into m Hamiltonian cycles $I_1, \ldots, I_m$ (see e.g. [3, Section 2.3]), each being an independent set in $\mathcal{M}$ by the definition.

By [1, Theorem 2], if $q \geq m-1$ then Breaker has the required winning strategy. (Here is a very brief sketch: whenever Maker selects $e \in I_i$, Breaker removes an edge from each I_j with $j \neq i$, so that in $\mathcal{M}' = \mathcal{M}/e$, the matroid with the edge e contracted, the updated sets $I'_1, \ldots, I'_m$ are still independent.) ∎

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