Constructions of Non-Principal Families in Extremal Hypergraph Theory

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For Miklós Simonovits on his 60th birthday

Abstract

A family $\mathcal{F}$ of k-graphs is called *non-principal* if its Turán density is strictly smaller than that of each individual member. For each $k \geq 3$ we find two (explicit) k-graphs F and G such that $\{F, G\}$ is non-principal. Our proofs use stability results for hypergraphs. This completely settles the question posed by the first author and Rödl [18].

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Also, we observe that the demonstrated non-principality phenomenon holds also with respect to the Ramsey-Turán density as well.

1 Introduction

In this paper, we prove the non-principality phenomenon for the classical extremal problems for k-uniform hypergraphs. The main motivation is to study the qualitative difference between the cases $k = 2$, and $k \geq 3$, and our results for the Turán problem exhibit this difference. We also study this question in the context of Ramsey-Turán theory, introduced by Erdős and Sós. Although we prove the non-principality phenomenon for Ramsey-Turán problems when $k \geq 3$, the behavior for $k = 2$ remains open. This is one of the few cases where an extremal problem for hypergraphs can be solved but not for graphs.

Given a family $\mathcal{F}$ of k-uniform hypergraphs (*k-graphs* for short), let

$$\mathrm{ex}(n, \mathcal{F}) := \max\{\, |G| : v(G) = n,\ \mathcal{F} \not\subset G \,\}$$

be the maximum size of a k-graph G on n vertices which is *$\mathcal{F}$-free* (that is, for every $F \in \mathcal{F}$ we have $F \not\subset G$). It was observed by Katona, Nemetz, and Simonovits [13] that the ratio $\mathrm{ex}(n, \mathcal{F})/\binom{n}{k}$ is non-increasing with n. In particular, the limit

$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}$$

exists; we call $\pi(\mathcal{F})$ the *Turán density* of $\mathcal{F}$. When $\mathcal{F} = \{F\}$ consists of a single forbidden k-graph, we write $\mathrm{ex}(n, F)$ and $\pi(F)$ for $\mathrm{ex}(n, \{F\})$ and $\pi(\{F\})$.

The first author and Rödl [18] conjectured that there is a family $\mathcal{F}$ of 3-graphs such that

$$\pi(\mathcal{F}) < \min\{\pi(F) : F \in \mathcal{F}\}, \tag{1}$$

and commented that the result should even hold for a family $\mathcal{F}$ of size two. Balogh [1] proved the conjecture, calling this phenomenon the *non-principality* of the Turán function. This is in sharp contrast with the case of graphs ($k = 2$) where the Erdős-Stone-Simonovits Theorem [7, 4] applies, giving that

$$\pi(\mathcal{F}) = \min\left\{1 - \frac{1}{\chi(F) - 1} : F \in \mathcal{F}\right\} = \min\left\{\pi(F) : F \in \mathcal{F}\right\}.$$

Unfortunately, the family in [1] has many members, many more than two. In Section 2 we present a new approach which shows how the so-called stability results lead to families $\mathcal{F}$ satisfying (1) and consisting of two k-graphs only. Combined with the authors' recent results [16, 19] this approach allows us to prove that for every $k \geq 3$, there is a non-principal k-graph family $\mathcal{F}$ with $|\mathcal{F}| = 2$, thus completely answering the question in [18] (see also Balogh [1, p. 177]).

In Section 3 we show how to extend the ideas of Balogh [1] to arbitrary k-graphs, $k \geq 3$. Although this seems to give non-principal families having many elements (a result weaker than that in Section 2), this method is very simple and self-contained. So we include it too.

Many of the (conjectured) extremal examples for (hyper)graph Turán problems have large independent sets. Motivated by this observation, Erdős and Sós [5] restricted the underlying k-graphs in this problem, by requiring that they have no large independent sets. This new class of problems has become known as the *Ramsey-Turán problems*. More precisely, for $0 < \delta \leq 1$,

$$\mathrm{ex}(n, \mathcal{F}, \delta) = \max\left\{|G| : G \subset \binom{[n]}{k} \text{ s.t. } G \text{ is } \mathcal{F}\text{-free and } \alpha(G) < \delta n\right\},$$

or zero if no such hypergraph exists. The *Ramsey-Turán density* $\rho(\mathcal{F})$ is defined as

$$\sup_{\delta(n)}\left\{\limsup_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F}, \delta(n))}{\binom{n}{k}} : \delta(n) \to 0 \text{ as } n \to \infty\right\}.$$

Very little is known about the parameter ρ for k-graphs. Indeed, computing $\rho(\mathcal{F})$ seems even more difficult than computing $\pi(\mathcal{F})$. For example, it is unknown whether $\rho(K_{2,2,2})$ is 0, where $K_{2,2,2}$ is the complete 3-partite graph with two vertices in each part.

Based on the complexity of determining ρ, one would suspect that ρ is also non-principal in the sense of this paper. Indeed, this is the case. Let $\mathcal{F}$ be any k-graph family satisfying (1). Let $\mathcal{F}' = \{F(2) : F \in \mathcal{F}\}$ be obtained from $\mathcal{F}$ by blowing-up each member with factor 2 (that is, each vertex is cloned once). Erdős and Sós [6] proved that if for any edge $D \in F$ there is another edge $D' \in F$ with $|D \cap D'| \geq 2$, then $\rho(F) = \pi(F)$. This result extends easily to families of k-graphs, yielding that $\rho(\mathcal{F}') = \pi(\mathcal{F}')$ and $\rho(F) = \pi(F)$ for any $F \in \mathcal{F}'$. By the supersaturation result of Erdős and Simonovits [2], blow-ups preserve the Turán density: $\pi(\mathcal{F}) = \pi(\mathcal{F}')$. Hence, $\mathcal{F}'$ is non-principal with respect to both the Turán and Ramsey-Turán densities.

Curiously, the situation with graphs remains open.

Problem 1 *Do there exist graphs G_1, G_2 for which*

$$\rho(\{G_1, G_2\}) < \min\{\rho(G_1), \rho(G_2)\}?$$

What about if we require $\rho(\{G_1, G_2\}) > 0$ as well?

2 Non-Principal Families of Size 2

We need some preliminary definitions before we can start proving the claimed results.

To obtain the *cone* $\mathrm{cn}(F)$ of a k-graph F, enlarge each edge of F by a new common vertex x:

$$\mathrm{cn}(F) := \{\{x\} \cup D : D \in F\}.$$

Let $\mathrm{cn}^{(i)}(F)$ be obtained from F by iterating the cone-operation i times. Define $\mathrm{cn}(\mathcal{F}) := \{\mathrm{cn}(F) : F \in \mathcal{F}\}$. Let K_m^k be the complete k-graph on m vertices.

We call two order-n k-graphs F and G *ε-close* if there is a bijection $f : V(F) \to V(G)$ between their vertex sets such that the number of k-subsets $D \subset V(F)$ for which $D \in F \not\Leftrightarrow f(D) \in G$ is at most $\varepsilon\binom{n}{k}$. In other words, we can make F isomorphic to G by adding and removing at most $\varepsilon\binom{n}{k}$ edges.

A k-graph G is *F-extremal* if it is a maximum F-free k-graph of order $v(G)$ (that is, $|G| = \mathrm{ex}(v(G), F)$). Let us call a k-graph F *stable* if for any $\delta > 0$ there are $\varepsilon > 0$ and n_0 such that for $n \geq n_0$ any F-free k-graph G of order n with at least $(\pi(F) - \varepsilon)\binom{n}{k}$ edges is δ-close to an F-extremal k-graph. Erdős [3] and Simonovits [20] independently proved that every 2-graph is stable.

The following lemma gives us a new approach to generating non-principal families.

Lemma 2 *Let F be a stable k-graph. Suppose that we can find a k-graph H of order h and a constant $c = c(H)$ such that:*

1. *$\pi(H) \geq \pi(F)$;*
2. *Any F-extremal k-graph of order $n \geq n_0$ contains at least cn^h copies of H.*

Then

$$\pi(\{F, H\}) < \min(\pi(F), \pi(H)). \tag{2}$$

Proof. Let $\varepsilon > 0$ and n_0 be the constants satisfying the stability assumption for F with $\delta = c$.

Suppose on the contrary that $\pi(\{F, H\}) \geq \pi(F)$. Then there is a k-graph G_n on $n \geq n_0$ vertices such that

1. G_n is $\{F, H\}$-free;
2. $|G_n| > (\pi(F) - \varepsilon)\binom{n}{k}$.

By the definition of ε and n_0, this k-graph G_n is c-close to an F-extremal k-graph G'_n on n vertices. By hypothesis, G'_n contains at least cn^h copies of H. Each edge of G'_n lies in at most $k!\, n^{h-k}$ copies of H. Consequently, in order to delete all copies of H from G'_n, we need to delete at least $\frac{cn^h}{k!\, n^{h-k}} > c\binom{n}{k}$ edges from G'_n. This contradicts the fact that G_n is H-free and is c-close to G'_n. ▮

Theorem 3 *For every $k \geq 3$ there are k-graphs F and H satisfying (2).*

Proof. Let the k-graph H_l^k be obtained from the complete 2-graph K_l^2 by enlarging each edge by a set of $k-2$ new vertices. Let $T_k(n,l)$, $k \leq l$, be the k-graph obtained by partitioning $[n] = V_1 \cup \ldots \cup V_l$ into l almost equal parts and taking those k-sets which intersect every part in at most one vertex.

The first author [16] proved that for arbitrary $l \geq k \geq 3$ we have

$$\pi(H_{l+1}^k) = \frac{l(l-1)\ldots(l-k+1)}{l^k}. \qquad (3)$$

The second author [19], building upon the results in [16], proved that H_{l+1}^k is stable with $T_k(n,l)$ being the unique maximum H_{l+1}^k-free graph of order n for $l \geq k \geq 3$ and $n \geq n_0(k,l)$.

Observe that $\pi(K_{k+1}^k) \geq \frac{3}{8} \geq \pi(H_{k+2}^k)$ for all $k \geq 3$. The lower bound on $\pi(K_{k+1}^k)$ follows from the following construction. Partition $[n] = A \cup B$ into two almost equal parts. Split the family of all k-sets X intersecting both A and B into two k-graphs G_0 and G_1 according to the parity of $|X \cap A|$. Both G_0 and G_1 are K_{k+1}^k-free: for any $(k+1)$-set $Y \subset [n]$ intersecting A in $s \in [1,k]$ elements, the k-sets $Y \setminus \{a\}$ and $Y \setminus \{b\}$, where $a \in Y \cap A$ and $b \in Y \cap B$, have the intersections with A of sizes $s-1$ and s respectively, that is, of different parities. Now, $|G_0| + |G_1| = (1 - 2^{-k+1} + o(1))\binom{n}{k}$, so one of these has size at least $\frac{1}{2}(1 - 2^{-k+1} + o(1))\binom{n}{k} \geq (\frac{3}{8} + o(1))\binom{n}{k}$. Consequently, $\pi(K_{k+1}^k) \geq \frac{3}{8}$. On the other hand, the Turán density $\pi(H_{k+2}^k) = \frac{(k+1)!}{(k+1)^k}$ is a decreasing function of k which equals $\frac{3}{8}$ for $k = 3$.

Finally, note that $T_k(n, k+1)$ contains at least $(\lfloor \frac{n}{k+1} \rfloor)^{k+1}$ copies of K_{k+1}^k. Lemma 2, whose all assumptions are satisfied, implies that the pair $\{H_{k+2}^k, K_{k+1}^k\}$ is non-principal for all $k \geq 3$. ∎

2.1 Using Other Stability Results

Although the paper [19] was accepted by the *Journal of Combinatorial Theory, Series B*, its publication is suspended because of a disagreement between the author and the publisher over the copyright terms[1]. So, it might be useful (and of independent interest) to see for which k we can infer the conclusion of Theorem 3 without referring to [19].

In a recent manuscript [17] we present a self-contained solution of the Turán problem for the *generalized fan* F_l^k which is a k-graph closely related to H_l^k. More precisely, the edge set of F_l^k comprises $[k]$ together with $E_{ij} \cup \{i, j\}$ over all pairs $\{i, j\} \in \binom{[l]}{2} \setminus \binom{[k]}{2}$, where E_{ij} are pairwise disjoint $(k-2)$-sets consisting of vertices outside $[l]$. It was proved that, for $l \geq k \geq 3$ and all large n, the k-graph F_{l+1}^k is stable and $T_k(n, l)$ is the unique extremal F_{l+1}^k-free graph. Since the same holds if we forbid H_{l+1}^k, the proof of Theorem 3, with obvious modifications, shows that for any $k \geq 3$, we have

$$\pi\left(\{F_{k+2}^k, K_{k+1}^k\}\right) < \min\left(\pi(F_{k+2}^k),\, \pi(K_{k+1}^k)\right).$$

The k-graph F_{k+2}^k is in a sense simpler than H_{k+2}^k, having only $2k+2$ edges when compared to $|H_{k+2}^k| = \binom{k+2}{2}$.

Let $k = 2l \geq 4$ be even. Let $F = \{A \cup B, A \cup C, B \cup C\}$, where A, B, C are disjoint l-sets. Frankl [8] showed that $\pi(F) = \frac{1}{2}$. Keevash and Sudakov [15, Theorem 3.4] showed that F is stable. Every extremal k-graph G' for F on $n \geq n_0$ vertices has vertex partition $X \cup Y$, $|X| \approx |Y| \approx \frac{n}{2}$, and consists of all edges intersecting X (and also Y) in an odd number of vertices. Let us take $H = \mathrm{cn}(K_m^{k-1})$ where $m = m(k)$ is a sufficiently large integer to satisfy

[1] See `http://www.math.cmu.edu/~pikhurko/Copyright.html` for more details.

$\frac{k!}{m^k}\binom{m}{k} > \frac{1}{2}$. The latter implies that $\pi(H) > \frac{1}{2}$, because $T_k(n,m)$ does not contain H. As G' contains $(2+o(1))\,\frac{n}{2}\binom{n/2}{m}$ copies of H, Lemma 2 implies that the family $\{F, H\}$ is non-principal.

For $k = 3$ we can use the stability result either for the Fano plane, established independently by Füredi and Simonovits [12] and by Keevash and Sudakov [14], or for

$$F_{3,2} := \{\,\{1,2,3\},\ \{1,4,5\},\ \{2,4,5\},\ \{3,4,5\}\,\}$$

established by Füredi, Simonovits, and the second author [11] (cf. also [10]). In both cases we can take $H = \mathrm{cn}(K_m^2)$ for some sufficiently large m.

3 Balogh's Construction for General $k \geq 3$

A partition $V(F) = \cup_{i=1}^m A_i$ of the vertex set of a k-graph F is called a $(k_1, \ldots, k_m)$*-partition* if every edge of F intersects A_i in precisely k_i vertices, $i \in [m]$. Let $\mathcal{N}_{k,l}$ be the (infinite) family consisting of all k-graphs not admitting a $(k-l, l)$-partition. Let the k-graph $T_k := \mathrm{cn}^{(k-2)}(K_3^2)$, where the operation $\mathrm{cn}(F)$ is defined in Section 2.

Theorem 4 *For every $k \geq 3$ there exists a finite family $\mathcal{F}$ of k-graphs which satisfies (1).*

Proof. Our construction generalizes that of Balogh [1]. We consider first the following (infinite) family

$$\mathcal{H} = \mathcal{H}_k := \{T_k\} \cup \bigcup_{i=0}^{k-3} \mathrm{cn}^{(i)}(\mathcal{N}_{k-i,1}).$$

We show that $\mathcal{H}$ satisfies (1) (with min replaced by inf) and then explain how to obtain the required finite $\mathcal{F}$ from it.

Let G be any $\mathcal{H}$-free k-graph of order n. We prove by induction on $k \geq 2$ that $|G| \leq p_k(n)$, where $p_k(n) := \prod_{i=1}^{k} \left\lfloor \frac{n+i-1}{k} \right\rfloor$ is the maximum size of a k-partite k-graph on n vertices.

The claim is true for $k = 2$, since in this case $\mathcal{H} = \{K_3^2\}$ and (a special case of) Turán's theorem applies. Let $k \geq 3$. As $\mathcal{N}_{k,1} \subset \mathcal{H}$, G admits a $(1, k-1)$-partition $A \cup B$. For any $x \in A$ the *link graph*

$$G_x := \{D \not\ni x : D \cup \{x\} \in G\}$$

is $\mathcal{H}_{k-1}$-free. Moreover, all edges of G_x lie inside B by the definition of $A \cup B$. By the induction assumption, $|G_x| \leq p_{k-1}(b)$, where $b := |B|$. As each edge intersects A in precisely one vertex, we have

$$|G| = \sum_{x \in A} |G_x| \leq (n-b)\, p_{k-1}(b) \leq p_k(n),$$

proving the claim.

Thus $\pi(\mathcal{H}) \leq \frac{k!}{k^k}$. In fact, we have equality here as k-partite k-graphs demonstrate.

On the other hand, a maximum k-graph G of order n with a $(2, 1, \ldots, 1)$-partition has about $(\frac{n}{k})^{k-2}\binom{2n/k}{2} \approx 2\, \frac{k!}{k^k} \binom{n}{k}$ edges and is $\mathcal{H} \setminus \{T_k\}$-free. Also, by taking a maximum k-partite k-graph and replacing the last three parts by the T_3-free 3-graph of density $\frac{2}{7}$ constructed by Frankl and Füredi [9] we add $\Omega(n^k)$ edges (note that $p_3(n)/\binom{n}{3} \approx \frac{2}{9} < \frac{2}{7}$). A routine analysis shows that the constructed k-graph is T_k-free.

Hence, $\pi(F) \geq \frac{9}{7}\, \pi(\mathcal{H})$ for every $F \in \mathcal{H}$. There is an n_0 such that, for example, $\mathrm{ex}(n_0, \mathcal{H})/\binom{n_0}{k} \leq \frac{8}{7}\, \pi(\mathcal{H})$. As $\mathrm{ex}(n, \mathcal{H})/\binom{n}{k}$ is non-increasing, see [13], $\pi(\mathcal{F}) \leq \frac{8}{7}\, \pi(\mathcal{H})$, where $\mathcal{F}$ consists of all k-graphs from $\mathcal{H}$ with at most n_0 vertices. The obtained (finite) family $\mathcal{F}$ has clearly all the required properties. ∎

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