Minimizing the Number of Partial Matchings in Bipartite Graphs

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Abstract

Suppose that we know the vertex degrees in one part of a bipartite graph G. We compute the smallest number of matchings of size m that G can have (provided there is at least one). In fact, our results also apply to the more general problem of counting matchings in matroids.

1 Introduction

Let G be a bipartite graph with a bipartition $V(G) = X \cup Y$. Let $X := \{x_1, \ldots, x_n\}$. Let $d_i := d(x_i)$ be the degree of x_i. Here we solve the following problem.

Problem 1 *Given* $\mathbf{d} := (d_1, \ldots, d_n)$ *and an integer* $m \leq n$*, what is the smallest number of matchings of size* m *that* G *can have, provided there is at least one* m*-matching?*

Ostrand [4] (see Hwang [2] for another proof) has settled the case $m = n$ when the matchings to count must contain every vertex of X. McCarthy [3] generalized Ostrand's results to the setting where we have a matroid on Y and we count the number of *independent* n*-matchings*, that is, we additionally require that the set of matched vertices of Y is an independent set. Our bound on partial matchings holds also for matroids, see Section 4.

One motivation behind this study is that sometimes the existence of a certain combinatorial object can be proved by applying Hall's Marriage Theorem (see [1, Chapter VIII.2] for some examples). Thus a lower bound in Problem 1 should give a quantitative strengthening of these results wherein we deduce a lower bound on the number of the constructed objects.

2 Notation and Preliminary Remarks

When dealing with matroids we will follow the terminology in [5]. Given a matroid $\mathcal{M}$ on Y, let $I_m(G,\mathcal{M})$ denote the number of independent m-matchings. Rado's theorem [6] implies that

$$(G,\mathcal{M}) \text{ has an independent } m\text{-matching iff } \forall A\subseteq X \ \rho(\Gamma(A)) \geq |A|-n+m, \tag{1}$$

where ρ is the rank function of $\mathcal{M}$ and $\Gamma(A)=\big\{y : \exists x\in A\ \{x,y\}\in E(G)\big\}$.

Any set A achieving the bound in (1) is called *critical.* It is easy to see that for any critical A every independent m-matching contains $\rho(\Gamma(A))$ vertices from $\Gamma(A)$ (the largest possible number) as well as all vertices in $X\setminus A$ but does not connect these two sets.

If $\mathcal{M}$ is the free matroid (that is, $\rho(A)=|A|$ for all $A\subseteq Y$), then (1) gives the well-known defect version of Hall's marriage theorem.

Note that m-matchings in G can be equivalently considered as systems of m distinct representatives of the set system $(\Gamma(x_1),\ldots,\Gamma(x_n))$. However, in this paper we will use the graph version.

3 Construction and Its Properties

First of all, we can assume without loss of generality that each d_i is positive (otherwise we remove x_i) and that $d_1\leq\cdots\leq d_n$.

To construct our graph $H=H_m(\mathbf{d})$ we have to specify sets $\Gamma(x_i)$. Let us assume that Y is an initial segment of positive integers (and $\mathcal{M}$ is the free matroid). For $i\in[n]$ define

$$\Gamma(x_i):=\begin{cases}[d_i], & \text{if } d_i\geq i-n+m,\\ [d_i-1]\cup\{i-n+m\}, & \text{otherwise.}\end{cases}$$

Note that H contains a matching of size m: consider the edges $\{x_i, i-n+m\}$ for $i\in[n-m+1,n]$.

Let us state a few properties of H which we will need later. Let $X_i:=\{x_1,\ldots,x_i\}$.

Lemma 2 *If we have $d_i\leq i-n+m$ for some i, then $\Gamma(X_i)=[i-n+m]$. (In particular, X_i is critical and H has no matching of size $m+1$.)*

Proof. For any $j\leq i$ we have $d_j\leq d_i\leq i-n+m$, so $\Gamma(x_j)\subseteq[i-n+m]$, proving $\Gamma(X_i)\subseteq[i-n+m]$. The converse inclusion follows by observing that $j\in[m]$ is always connected to x_{j+n-m}. ∎

Lemma 2 allows us to compute $f_m(\mathbf{d})$, the number of m-matchings in H.

If $d_i \leq i - n + m$ for some i, then

$$f_m(\mathbf{d}) = \frac{1}{(n-m)!} \prod_{i=1}^{n} \max(d_i + n - m - i + 1, 1). \tag{2}$$

Indeed, if we add $n - m$ new vertices to Y which are connected to everything in X, then, in view of Lemma 2, the new graph H' has precisely $(n-m)! \cdot f_m(\mathbf{d})$ matchings of size n. Note that $H' \cong H_n(d_1 + n - m, \ldots, d_n + n - m)$ and for this graph it is easy to compute the number of n-matchings (alternatively, see Ostrand [4]), giving (2).

If $d_i > i - n + m$ for all i, then we have $\Gamma(x_i) \subseteq \Gamma(x_j)$ for any $i < j$ and the number of m-matchings can be expressed as

$$f_m(\mathbf{d}) = \sum_{1 \leq \nu_1 < \cdots < \nu_m \leq n} \prod_{i=1}^{m} \max(d_{\nu_i} - i + 1, 0). \tag{3}$$

It seems that there is no nice formula, like (2), for $f_m(\mathbf{d})$ in this case.

In the remainder of this paper, when we write $f_m(\mathbf{d})$ we will mean that we remove any zeros from $\mathbf{d}$, reorder $\mathbf{d}$ to be non-decreasing and then use the formulas (2) and (3).

Lemma 3 *The function $f_m(\mathbf{d})$ is non-decreasing with respect to each argument d_i.*

Proof. It is enough to prove the claim when we increase some d_i by 1: $d_i' = d_i + 1$ while all other $d_j' = d_j$. We can assume that either $i = n$ or $d_i < d_{i+1}$. When we analyze the corresponding graphs, H and H', we see that H' is obtained from H by adding one more edge. Of course, this cannot decrease the number of m-matchings. ∎

4 Lower Bound

In this section the term 'matching' implicitly means 'an independent matching.'

Theorem 4 *Let G be a bipartite graph with a bipartition $V(G) = X \cup Y$. Let $\mathcal{M}$ be a matroid on Y with rank function ρ. Let $X := \{x_1, \ldots, x_n\}$ and $d_i := \rho(\Gamma(x_i))$. Assume $1 \leq d_1 \leq \cdots \leq d_n$. If $I_m(G, \mathcal{M}) \geq 1$, then*

$$I_m(G, \mathcal{M}) \geq f_m(d_1, \ldots, d_n). \tag{4}$$

Proof. We use induction on n with the case $n = 1$ being trivially true. Let $n \geq 2$. The proof splits into two cases. Recall that a set $A \subseteq X$ is called critical if we have equality in (1).

Case 1 There is a critical $A \subseteq X$ (possibly $A = X$).

This means that $(G, \mathcal{M})$ admits no $(m+1)$-matching. Let G' be obtained from G by adding $n-m$ new vertices to Y which are connected to everything in X. Let the matroid $\mathcal{M}'$ be the matroid union of $\mathcal{M}$ and the free matroid on the new vertices; its rank function is

$$\rho'(B) = \rho(B \cap Y) + |B \setminus Y|.$$

Clearly, $I_n(G, \mathcal{M}) = I_m(G', \mathcal{M}')/(n-m)!$. Now, the result of McCarthy [3], when applied to $(G', \mathcal{M}')$, settles this case.

Case 2 There is no critical set.

Let us bound N_1, the number of m-matchings containing x_1. We can choose a non-loop $y \in \Gamma(x_1)$ in at least d_1 possible ways.

Let us show that the pair $(G', \mathcal{M}')$, where $G' := G - x_1 - y$ and $\mathcal{M}' := \mathcal{M}/y$, has an $(m-1)$-matching. If this is not true, then by (1) we can find $A \subseteq X \setminus \{x_1\}$ with

$$\rho'(\Gamma_{G'}(A)) \leq |A| - (n-1) + (m-1) - 1 = |A| - n + m - 1.$$

This implies that A is critical with respect to $(G, \mathcal{M})$, a contradiction.

Clearly, $\rho'(\Gamma_{G'}(x_i)) \geq d_i - 1$. By the monotonicity of f_m and induction on n, we have

$$N_1 \geq d_1 f_{m-1}(d_2 - 1, \dots, d_n - 1). \tag{5}$$

To bound N_2, the number of m-matchings omitting x_1, let $G' := G - x_1$. Similarly to above, one can show that $(G', \mathcal{M})$ has an m-matching. Thus

$$N_2 \geq f_m(d_2, \dots, d_n). \tag{6}$$

To complete the proof, it is enough to prove that

$$f_m(d_1, \dots, d_n) \leq d_1 f_{m-1}(d_2 - 1, \dots, d_n - 1) + f_m(d_2, \dots, d_n). \tag{7}$$

If the value d_1 occurs in $\mathbf{d}$ at most $d_1 + n - m$ times, then in $H_m(\mathbf{d})$ we have $\Gamma(x_1) \subseteq \Gamma(x_i)$ for any i. Splitting m-matchings of $H_m(\mathbf{d})$ into two groups according to whether or not they contain x_1 we conclude that (7) holds. (It is an equality, in fact.)

So, suppose that d_1 appears $j > d_1 + n - m$ times in $\mathbf{d}$: $d_1 = \cdots = d_j$. Here we deduce first that

$$f_m(\mathbf{d}) \leq d_1 f_{m-1}(\mathbf{d}') + f_m(d_2, \dots, d_n), \tag{8}$$

where $\mathbf{d}'$ consists of $d_1 - 1$ repeated $d_1 + n - m - 1$ times, then d_1 repeated $j - d_1 - n + m$ times, followed by $d_{j+1} - 1, \dots, d_n - 1$. But in $H_{m-1}(\mathbf{d}')$ the vertices of degree $d_1 - 1$ form a critical set by Lemma 2 so they claim the whole of $[d_1 - 1]$ in any $(m-1)$-matching. The graph $H_{m-1}(\mathbf{d}')$ is obtained from $H_{m-1}(d_2 - 1, \ldots, d_n - 1)$ by adding extra edges connecting $[d_1 - 1] \subseteq Y$ to degree-d_1 vertices in X. This shows that

$$f_{m-1}(\mathbf{d}') = f_{m-1}(d_2 - 1, \ldots, d_n - 1)$$

and implies (7) by (8), finishing the proof. ∎

5 Concluding Remarks

Observe that Problem 1 can also be solved if we omit the condition that G contains an m-matching. Indeed, it is straightforward to deduce from (1) that the restrictions on $\mathbf{d}, n$ force an m-matching if and only if $d_i \geq i - n + m$ for each $i \in [n]$.

The question of *maximizing* the number of m-matchings is trivial with the extremal construction being the disjoint union of stars K_{1,d_i}. (While for matroids there is no upper bound at all.)

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