# OPTIMIZERS FOR SUB-SUMS SUBJECTED TO A SUM- AND A SCHUR CONVEX CONSTRAINT WITH APPLICATIONS TO THE ESTIMATION OF EIGENVALUES

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Running title:

Optimizers for Sub-Sums and Eigenvalue Estimation

Abstract: A complete solution is presented for the problem of determining the sets of points at which the functions $(x_1, \ldots, x_n) \mapsto x_k + \ldots + x_l$, subject to the constraints $M \geq x_1 \geq \ldots \geq x_n \geq m$, $x_1 + x_2 + \ldots + x_n = a$, and $g(x_1) + g(x_2) + \ldots + g(x_n) = b$, with $g$ strictly convex continuous, assume their maxima and minima. Applications are given.

MSC: Primary: 90C25 Convex programming; Secondary: 26D15 Inequalities for Sums, Series, Integrals; 15A42 Inequalities involving eigenvalues.

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# Optimizers for Sub Sums subjected to a Sum- and a Schur Convex Constraint with Applications to the Estimation of Eigenvalues

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## 0. Introduction

Let $m, M \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, $a, b \in \mathbb{R}$, and $n \in \mathbb{Z}_{\geq 1}$. For an extended function $g : \mathbb{R} \to ]-\infty, +\infty]$, and integers $1 \leq k \leq l \leq n$, define $S_g^{kl}(\mathbf{x}) = \sum_{i=k}^{l} g(x_i)$. We also use the abbreviations $S_g = S_g^{1n}$, $S^{kl} = S_{\mathrm{id}}^{kl}$, and $S = S^{1n}$. For convex continuous $g : \mathbb{R} \to ]-\infty, +\infty]$ define the spaces

$$X = X(m, M, a, b; n; g) = \{x \in \mathbb{R}^n : M \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq m, S(x) = a, S_g(x) = b\}.$$

The main objective of this paper is the determination of the subsets of $X$ where the functions $S^{kl}$ assume their maxima and minima (called also maximum sets or maximizers, and minimum sets or minimizers) via an elementary technique; i.e. avoiding Karush-Kuhn-Tucker theory; see e.g. [BL]. The calculation of upper and lower bounds for the functions $S^{kl}$, of necessity always the best possible, is then a simple matter of evaluation of the functions at points in their maximizers and minimizers.

This article puts many of the results estimating various functions of eigenvalues and singular values in terms of the trace and/or determinant etc. e.g. [GJSW], [MST], [MSW], [MUV], [MUVTU], [MV],[MW],[WS1],[WS2]; in particular those in the last four papers mentioned under a general umbrella; all the bounds obtained by Wolkowicz and Styan [WS1],[WS2] are implied by our results.

Section 1 recalls some basics of the modern theory of convex functions; facts yielding information about the spaces $X(m, M, *)$, in particular also for $n = 3$ via generalized barycentric coordinates in the plane, are also proved. In section 2, after presenting in lemma 6 the heart of our technique and theorem 7 that implicitly contains an algorithm to test when the spaces $X$ are nonempty, we show that for all integers $1 \leq k \leq l \leq n$ and $m, M \in \overline{\mathbb{R}}$ the sets minimizer$(S^{kl}) = \{\mathbf{x} \in X : S^{kl}(\mathbf{x}) = \inf\{S^{kl}(\mathbf{u}) : \mathbf{u} \in X\}\}$ and maximizer$(S^{kl}) = \{\mathbf{x} \in X : S^{kl}(\mathbf{x}) = \sup\{S^{kl}(\mathbf{u}) : \mathbf{u} \in X\}\}$ consist of elements having a simple structure. Indeed theorems 8 and 9 show that the minimization or maximization of the functions $S^{kl}$ can be reduced to that of solving equations in two unknowns or to that of finding the intersections of a plane convex curve with a straight line. The extent to which these equations can be explicitly solved depends of course on the convex function $g$. The brief section 3 indicates that our theorems allow to calculate explicitly the infimum and supremum of certain subsets of $X$ in the lattice induced by majorization order on $X$. In section 4 we show how our results can be used to estimate partial sums of the eigenvalues of matrices having real spectra, given additional informations on the matrix like trace$A$ and trace$A^2$; trace$A$ and trace$A^3$; trace$A$ and $\det A$, etc. This way we gain deeper understanding of results found earlier by the first two and other authors on this subject.

Concerning presentation, we make differentiability assumptions on the convex function $g$. These are made for simplicity, since they do not impose restrictions on our applications, since they broaden accessibility of the paper, and since at spots they lead to further illuminating observations. As far as we see the main results theorem 8,9 can be stated and proved without differentiability assumptions following the indications in the concluding section 5. Though the basic idea of our method is simple, we apologize for that there are sometimes many cases to consider, the number and implications of which are at times best controlled by applying a modest amount of the symbology of logic. Our objects frequently depend on many parameters. To lighten notation, we usually suppress a fair amount of these; the context provides the remaining ones. Finally, for precision we indicate references sometimes in ways like '[B, p12c-5]' meaning 'see [B] page 12, approximately centimeter 5 from last text row'.

## 1. Some Notation and Basic Definitions and Lemmas

a. Some of the symbols and conventions we use are these:

| | |
|---|---|
| $\mathbb{R}^n$ | $n$-dimensional affine space; also space of real $n$-tuples. Sometimes replaced by a copy considered as an abstract Euclidean space $E$. |
| $\mathbf{1}_n$ | the n-tuple $(1,1,\ldots,1) \in \mathbb{R}^n$. |
| $\mathbf{x}$; $\mathbf{x}(k:l)$ | $n$-tuple $(x_1,\ldots,x_n)$; for integers $1 \le k,l \le n$, the subtuple $(x_k,x_{k+1},\ldots,x_l)$; empty if $k>l$. (The notation is MATLAB-inspired.) |
| $\mathbf{x}_\downarrow$ | decreasing rearrangement of $\mathbf{x}$; e. g. $(1,3,4,2)_\downarrow = (4,3,2,1)$. |
| $[\mathbf{p},\mathbf{q}]$ | a segment connecting points $\mathbf{p},\mathbf{q}$ in affine space. |
| $\vee,\wedge,\exists$ | logical symbols for non-exclusive 'or', 'and', 'exists'. |
| $P(m,M,a;n)$ | the set $\{\mathbf{x} \in \mathbb{R}^n : M \ge x_1 \ldots \ge x_n \ge m, \sum_{i=1}^n x_i = a\}$ |
| $\mathcal{D}(n;a)$ | the set of all decreasing $n$-tuples of sum $a$; same as $P(-\infty,+\infty,a;n)$. |
| $*$ | sometimes used for arguments we do not wish to specify. E.g: 'consider a space $X(m,M,*)$' is ' consider some space $X(m,M,a,b,n;g)$'. The notation can mean also a union. E.g. 'for any $\mathbf{x},\mathbf{y} \in \mathcal{D}(n,*)$' can mean 'for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ that are decreasing'. Context will avoid confusion. |
| $\partial B$ | topological boundary of set $B$. |
| $B(\mathbf{p},\rho)$ | $\mathbf{p}-$centered ball of Euclidean radius $\rho$. |
| $\triangleright, \triangleleft$ | beginning and end of a smaller piece of reasoning; e.g. of a proof of a claim |
| $\mathbf{x} \le \mathbf{y}, \mathbf{x} > \mathbf{y}$, etc. | comparability of $n$-tuples in componentwise order. |
| $\mathbf{x} \preceq_w \mathbf{y}$; $\mathbf{x} \preceq \mathbf{y}$ | weak; strong majorization orders. Defined for $\mathbf{x},\mathbf{y} \in \mathcal{D}(n,*)$, saying $S^{1l}(\mathbf{y}-\mathbf{x}) \ge 0$ for $l=1,\ldots,n$; $\mathbf{x} \preceq \mathbf{y}$ means $\mathbf{x} \preceq_w \mathbf{y} \wedge S^{1n}(\mathbf{y}-\mathbf{x}) = 0$. Definitive treatise on $\preceq_w$ is [MO]. |
| $0 \cdot \pm\infty, a+\infty$ | by conventions $0 \cdot \pm\infty = 0, a+\infty = \infty$. |
| $\mathbf{p} \sim \mathbf{q}$ | proportionality, saying $\exists \lambda \ne 0, \mathbf{p} = \lambda \mathbf{q}$. |
| w.r.t., lhs$(\cdot)$,rhs$(\cdot)$ | with respect to, left-hand side of $(\cdot)$, right hand side of $(\cdot)$. |

b. Concerning convex functions it simplifies matters if we adopt definitions used in modern texts on convex analysis or optimization; foremost we mention Rockafellar's definitive treatise [R], and the recent book of Borwein and Lewis [BL]. Let $E$ be an Euclidean space and $g : E \to [-\infty,+\infty]$ be a real valued function on $E$. The *domain* of $g$ is $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$; $g$ is called *proper* if $\operatorname{dom} g \ne \emptyset$ and for no $x \in E$, $g(x) = -\infty$, see [BL, p44c2],[R, p23c4]. $g$ is *convex* if its *epigraph* $\operatorname{epi} g = \{(x,r) \in E \times \mathbb{R} : g(x) \le r\}$ is a convex set, see [BL, p43c-2], [R, p23]; strictly convex if its epigraph is strictly convex, see [L, p98c-2]. Proper convex functions in this sense have convex domain and satisfy for $x,y \in E$ the usual inequality $g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y)$, [BL, p46c6ff]; strictly convex functions the strict inequality. As we see, every convex/strictly convex function $g : I \to \mathbb{R}$ ($I$ is an interval) in the traditional sense (satisfying the inequality/strict inequality), can be trivially extended to $\mathbb{R}$ putting its values outside $I$ equal to $+\infty$ and thus viewed as a convex function in the sense here defined; see [R, p23c-3].

**Proposition 1** Let $k \in \mathbb{Z}_{\ge 1}$, $\alpha_1,\ldots,\alpha_k \in \mathbb{R}_{>0}$, $a,b \in \mathbb{R}$. Assume the function $g : \mathbb{R} \to [-\infty,+\infty]$ satisfies the following: $g$ is proper and strictly convex; $\operatorname{dom} g$ is open and $g$ is differentiable and unbounded on $\operatorname{dom} g$. Let $\mu = \frac{a}{\sum_{i=1}^k \alpha_i}$ and $\mathbf{m} = \mu \mathbf{1}_k$ and let $\mathcal{S} = \mathcal{S}((\alpha_1,\ldots,\alpha_k),a,b)$ be the set of solutions of the system of two equations

$$\text{i. } S(\mathbf{x}) = \sum_{i=1}^k \alpha_i x_i = a, \qquad \text{ii. } G(\mathbf{x}) = \sum_{i=1}^k \alpha_i g(x_i) = b.$$

Then $H := \{\mathbf{x} : \mathbf{x} \text{ solves i}\}$ is a hyperplane and $\mathcal{S}$ is either empty/the point $\{\mathbf{m}\}$/homeomorphic to a sphere in $H$ according to if $G(\mathbf{m}) > b/G(\mathbf{m}) = b/G(\mathbf{m}) < b$.

Proof. It is clear that $H$ is a hyperplane containing $\mathbf{m}$. We can write $H = \{\mathbf{m} + t\mathbf{r} : t, r_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i r_i = 0, ||\mathbf{r}|| = 1\}$. Every $\mathbf{x} \in H \setminus \{\mathbf{m}\}$ has a unique representation $\mathbf{x} = \mathbf{m} + t(\mathbf{x})\mathbf{r}(\mathbf{x})$ with $t(\mathbf{x}) > 0$, and thus we can view $G$ henceforth as defined by $G(\mathbf{m} + t\mathbf{r}) = \sum_{i=1}^{k} \alpha_i g(\mu + t r_i)$ as a function on Euclidean space $H$.

Claim 0. $G$ is strictly convex on H.
$\triangleright$Let $\mathbf{x}, \mathbf{y}$ be two points in $\operatorname{dom} G$ and $\lambda \in ]0,1[$. Then

$$\begin{aligned} G(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \sum_{i=1}^{k} \alpha_i g(\lambda x_i + (1-\lambda) y_i) \\ &\leq \sum_{i=1}^{k} \alpha_i (\lambda g(x_i) + (1-\lambda) g(y_i)) \\ &= \lambda G(\mathbf{x}) + (1-\lambda) G(\mathbf{y}). \end{aligned}$$

Since we are supposing $\mathbf{x} \neq \mathbf{y}$ there exists an $i$ such that $x_i \neq y_i$, hence the strict convexity of $g$ implies the strictness of the inequality.$\triangleleft$

Claim 1. $\operatorname{dom} G \neq \emptyset \Leftrightarrow \mathbf{m} \in \operatorname{dom} G \Leftrightarrow \mu \in \operatorname{dom} g$.
$\triangleright$The implications $\Leftarrow$ are clear. Now let $t \in \mathbb{R}$ be such that $\mathbf{m} + t\mathbf{r} \in \operatorname{dom} G$. There are $i, j$ such that $r_i \leq 0 \leq r_j$ and we get $g(\mu + t r_i) < +\infty$ and $g(\mu + t r_j) < +\infty$. If $t \geq 0$ we have $\mu + t r_i \leq \mu \leq \mu + t r_j$ and the reverse inequalities in case $t \leq 0$. In any case the convexity of $\operatorname{dom} g$ insures $\mu \in g$ as we wished to show.$\triangleleft$

Claim 2. $\operatorname{dom} G$ is open.
$\triangleright$If $\operatorname{dom} G$ is empty there is nothing to prove. Suppose now $\operatorname{dom} G \neq \emptyset$ and let $\mathbf{x} \in H$. Clearly $\mathbf{x} \in \operatorname{dom} G$ iff, for all $i$, $x_i \in \operatorname{dom} g$. Since $\operatorname{dom} g$ is open there is an $\varepsilon > 0$ such that $x_i + ]-\varepsilon, +\varepsilon[ \subseteq \operatorname{dom} g$ for all $i$. But then the open set $(\mathbf{x} + ]-\varepsilon, +\varepsilon[^n) \cap H$ in $H$ is an open neighborhood of $\mathbf{x}$ in $H$, contained in $\operatorname{dom} G$.$\triangleleft$

Claim 3. If $\operatorname{dom} G \neq \emptyset$, then $\operatorname{minimizer} G = \{\mathbf{m}\}$.
$\triangleright$By claims 1 and 2, $\mathbf{m}$ and $\mu$ are interior points of of $\operatorname{dom} G$ and $\operatorname{dom} g$ respectively. Hence the calculation $\frac{d}{dt}\tilde{G}(\mathbf{m} + t\mathbf{r})|_{t=0} = \sum_i \alpha_i r_i g'(\mu) = 0$ is well defined and shows that $\mathbf{m}$ is a critical point. Since $G$ is a strictly convex function, $\mathbf{m}$ is the unique minimizer of G, see [BL, p16c-7, 19c-5]. $\triangleleft$

Claim 4. For any $r \in \mathbb{R}$ the level sets $L_r = \{\mathbf{x} \in H : G(\mathbf{x}) \leq r\}$ are convex compact subsets of $\operatorname{dom} G$. If $r > G(\mathbf{m})$, then $\mathbf{m} \in \operatorname{int}(L_r) = \{\mathbf{x} : G(\mathbf{x}) < r\}$.
$\triangleright$We can assume $L_r \neq \emptyset$. By claims 2 and 3 there exists a $\rho > 0$ such that the sphere $\partial B$, $B = B(\mathbf{m}, \rho) \subset H$ pertains to $\operatorname{dom} G$. Let $d(\mathbf{r}) := G(\mathbf{m} + \rho\mathbf{r}) - G(\mathbf{m}) > 0$. The function $d$ assumes on $\partial B$ its minimum, say at point $\mathbf{r}_0$. From an application of the formulae in [RV, p98c1] or [R, p242c2] we get

$$\begin{aligned} G(\mathbf{m} + t\mathbf{r}) &\geq G(\mathbf{m} + \rho\mathbf{r}) + (t - \rho)\frac{dG}{dt}|_{t=\rho} \\ &\geq G(\mathbf{m}) + d(\mathbf{r}) + (t - \rho)\frac{d(\mathbf{r})}{\rho} \\ &= G(\mathbf{m}) + \frac{t}{\rho} d(\mathbf{r}) \\ &\geq G(\mathbf{m}) + \frac{t}{\rho} d(\mathbf{r}_0). \end{aligned}$$

Choose $t_0$ such that $G(\mathbf{m}) + \frac{t_0}{\rho} d(\mathbf{r}_0) = r$. We see: for all $\mathbf{x} \in H$ with $\mathbf{x} \notin B(\mathbf{m}, t_0)$, there holds $G(\mathbf{x}) > r$. Hence the set $L_r$ is bounded. The remaining claims follow from combining [R, p28c-2, 51c-3, 52c-8, 59c2] or can otherwise be left to the reader. $\triangleleft$

We can now conclude the proof. Case: $G(\mathbf{m}) > b$. If $\mathbf{m} \in \operatorname{dom} G$, then by claim 3, for all $\mathbf{x} \in H$, $G(\mathbf{x}) \geq G(\mathbf{m}) > b$. If $\mathbf{m} \notin \operatorname{dom} G$, then claim 1 says $G(\mathbf{x}) = +\infty$ for all $\mathbf{x} \in H$ and we come to the same conclusion. In any case we see $\mathcal{S} = \emptyset$. Case: $G(\mathbf{m}) = b$. Then claim 3 gives $\mathcal{S} = \{\mathbf{m}\}$. Case: $G(\mathbf{m}) < b$. Then using $\mathcal{S} = \{\mathbf{x} : G(\mathbf{x}) \leq b\} \setminus \{\mathbf{x} : G(\mathbf{x}) < b\}$, we have by [R, p59c1] that $\mathcal{S} = \partial L_b$. Now [B, corollary 11.3.4] concludes the proof. ∎

**Lemma 2** Assume the hypotheses of proposition 1 specialized to the case $k = 3$ and $G(\mathbf{m}) < b$. Then $\mathcal{S} = \mathcal{S}((\alpha, \beta, \gamma), a, b)$ is a convex, rectifiable, smooth curve in $H$. If $s \mapsto (x(s), y(s), z(s))$ is a regular parametric representation (e.g. w.r.t. curve length), then with a dot denoting differentiation w.r.t. $s$:
a. $\dot{x} = 0 \leftrightarrow z = y, \quad \dot{y} = 0 \leftrightarrow x = z, \quad \dot{z} = 0 \leftrightarrow x = y$;
b. for each $s$ with $\dot{x} \cdot \dot{y} \cdot \dot{z} \neq 0$ there holds $\operatorname{sgn}(\dot{x}, \dot{y}, \dot{z}) \in \{(+, -, +), (-, +, -)\}$;

Proof. It is known that the boundary of a compact convex plane region is rectifiable. Thus we obtain from the previous lemma that we can parametrize $\mathcal{S}$ in the form $s \mapsto (x(s), y(s), z(s))$, where $s$ is the arc length measured from a certain point onwards. (In a cartesian interpretation of coordinates, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \equiv 1$.) Since the curve $\mathcal{S}$ satisfies i and ii, we find upon differentiation the relations

$$\begin{aligned} 0 &= \alpha\dot{x} + \beta\dot{y} + \gamma\dot{z} && (1) \\ 0 &= \alpha g'(x)\dot{x} + \beta g'(y)\dot{y} + \gamma g'(z)\dot{z} \\ &= (\alpha\dot{x} + \beta\dot{y} + \gamma\dot{z})g'(z) + \alpha\dot{x}(g'(x) - g'(z)) + \beta\dot{y}(g'(y) - g'(z)) \\ &= \alpha\dot{x}(g'(x) - g'(z)) + \beta\dot{y}(g'(y) - g'(z)) && (2) \end{aligned}$$

Note that the nonvanishing of the tangent vector and (1) imply that for each $s$ at most one of the quantities $\dot{x}(s), \dot{y}(s), \dot{z}(s)$ vanishes. Thus if $\dot{x} = 0$ we find $g'(y) - g'(z) = 0$, hence, by monotonicity of $g'$, [RV, p10c-1], that $y = z$. Conversely, and using that $x = y = z$ is impossible, we get that $y = z$ implies $\dot{x} = 0$. Proceeding analogously for the pairs $x, z$ and $x, y$ proves (a). b. Assume $x > y > z$. Then $g'(x) > g'(y) > g'(z)$. Thus we see that $\operatorname{sgn}(\dot{x})\operatorname{sgn}(\dot{y}) = -1$. Interchanging in the calculation leading to (2) the roles of $\alpha g'(x)\dot{x}$ and $\gamma g'(z)\dot{z}$ we get similarly that $\operatorname{sgn}(\dot{x})\operatorname{sgn}(\dot{z}) = -1$. Similar considerations can obviously be made in all cases where $x \neq y \neq z \neq x$. ∎

It is instructive and for our later applications indeed useful to interpret the formulae of lemma 2 geometrically. Select three noncollinear points $X, Y, Z$ in a plane. We coordinatize its points by triples $(x, y, z)$ satisfying $\alpha x + \beta y + \gamma z = a$ via the better known barycentric coordinates $(\hat{x}, \hat{y}, \hat{z})$ w.r.t. $X, Y, Z$ (see [B, p81]) via the formulae $\hat{x} = \frac{\alpha}{a}x, \hat{y} = \frac{\beta}{a}y, \hat{z} = \frac{\gamma}{a}z$. We shall call $(x, y, z)$, generalized barycentric coordinates w.r.t. $X, Y, Z$ and parameters $\alpha, \beta, \gamma, a$.

**Lemma 3** In a generalized barycentric system with parameters $\alpha, \beta, \gamma, a$, an equation $cx + dy + ez = f$ with $(0, 0, 0) \not\sim (c, d, e) \not\sim (\alpha, \beta, \gamma)$ represents a line through points $Z' \in \{z = 0\}$ and $Y' \in \{y = 0\}$ (see the rhs-figure) located so that

$$\overrightarrow{XZ'} = \frac{\beta}{a} \cdot \frac{ac - \alpha f}{\beta c - \alpha d}\overrightarrow{XY}, \quad \overrightarrow{XY'} = \frac{\gamma}{a} \cdot \frac{ac - \alpha f}{\gamma c - \alpha e}\overrightarrow{XZ}.$$

Proof. Expressed in barycentric coordinates, $cx + dy + ez = f$ turns into $\frac{ac}{\alpha}\hat{x} + \frac{ad}{\beta}\hat{y} + \frac{ae}{\gamma}\hat{z} = f$. Given our hypotheses, this represents a line for it is interpretable as the intersection of two planes. Since $\hat{x} + \hat{y} + \hat{z} = 1$, one calculates that this line intersects $\{z = 0\} = \{\hat{z} = 0\}$ at a point $Z'$ whose barycentric coordinates are given by $\hat{x}(Z') = \frac{\alpha\beta f - \alpha a d}{\beta a c - \alpha a d}$, $\hat{y}(Z') = \frac{\beta a c - \alpha\beta f}{\beta a c - \alpha a d}$. One sees (e.g. applying [B, p295c-4] and elementary triangle geometry, or otherwise) that if $P$ is any point on the line $\hat{z} = 0$, then one has $\overrightarrow{XP} = \hat{y}(P) \cdot \overrightarrow{XY}$. ¿From this follows the first assertion; the second is similar. ∎

**Example 4** Consider the lhs-figure where we have chosen points $X, Y, Z$ to define a triangle of unit side length. In such a system the coefficients of $\overrightarrow{XY}$, $\overrightarrow{XZ}$ are interpretable as signed distances from $X$ to $Z'$ and $X$ to $Y'$ respectively.

The rest of the figure is to be interpreted with $(\alpha, \beta, \gamma, a) = (2, 5, 3, 1)$. The straight lines drawn are: prolonged segments $XY \subseteq \{z = 0\}$, $ZY \subseteq \{x = 0\}$, $ZX \subseteq \{y = 0\}$, that define the triangle; the cevians $x = y$ through $Z$, $x = z$ through $Y$, $z = y$ through $X$, and the line $3x+y = 0.5$. The figure shows further a convex curve which we assume to be the solution of some equation $2g(x) + 5g(y) + 3g(z) = b$ for a certain convex $g$. According to lemma 2, at the point of intersection with $x = y, y = z, z = x$ the tangents to $\mathcal{S}$ are parallel to segments $XY$, $YZ$ and $ZX$ respectively. We further find a shadowed open region in which the coordinates of any point satisfy $x > y > z$. There are 6 such regions defined solely by the lines through the point $x = y = z$. The arrows indicate increase or decrease of coordinates $x, y, z$ of a point moving along $\mathcal{S}$ in that region in counter clockwise direction. The lines $x = 0, y = 0, z = 0$ define 7 disjoint regions. The function $P \mapsto \mathrm{sgn}(x, y, z)$ is constant in each of these regions. We have indicated the values of this function for each of the regions by $-++$, etc. in the rhs-figure.

**Lemma 5** Assume the hypotheses and notation of lemma 2. Define for $m, M \in [-\infty, +\infty]$ the set $\mathcal{S}' = \mathcal{S}'(m, M) = \{(x, y, z) \in \mathcal{S}((\alpha, \beta, \gamma), a, b) : M \geq x \geq y \geq z \geq m\}$. Let $(\xi', \eta') \in \mathcal{S}((\alpha, \beta + \gamma), a, b)$ with $\xi' \geq \eta'$ and $(\xi, \eta) \in \mathcal{S}((\alpha + \beta, \gamma), a, b)$ with $\xi \geq \eta$. Let

$$(x_L, y_L, z_L) = \left\{ \begin{array}{rl} (\xi', \eta', \eta') & \text{if } \xi' \leq M \\ (M, y, z) \in \mathcal{S}' & \text{if } \xi' > M \end{array} \right. , \quad (x_R, y_R, z_R) = \left\{ \begin{array}{rl} (\xi, \xi, \eta) & \text{if } \eta \geq m \\ (x, y, m) \in \mathcal{S}' & \text{if } \eta < m \end{array} \right. .$$

These points are all uniquely defined. Assume $c, d, e \in \mathbb{R}_{\geq 0}$ satisfy in addition to the hypothesis of lemma 3 the inequality $\frac{c}{\alpha} > \max\{\frac{d}{\beta}, \frac{e}{\gamma}\}$. Put $\varepsilon = \mathrm{sgn}\left(\frac{\gamma}{\beta}\frac{\beta c-\alpha d}{\gamma c-\alpha e} + \frac{\alpha}{\beta}\frac{x_L-x_R}{y_R-y_L} - 1\right)$. Then

$$\text{minimizer}(cx+dy+ez|\mathcal{S}'(m,M)) = \begin{cases} \{(x_R,y_R,z_R)\} & \text{if } \varepsilon = +1 \\ \{(x_L,y_L,z_L),(x_R,y_R,z_R)\} & \text{if } \varepsilon = 0 \\ \{(x_L,y_L,z_L)\} & \text{if } \varepsilon = -1. \end{cases}$$

Proof. We reason along the lines of the figures. If $M$ is sufficiently large or $m$ sufficiently small, then $\mathcal{S}'(m,M)$ is just the part of $\mathcal{S}$ lying in the shadowed region $x \geq y \geq z$ of the lhs-figure and thus limited at the left by a point $(x,y,z)$ with $y = z$, coinciding evidently with the point $(\xi',\eta',\eta')$ formed from the solution $(\xi',\eta')$ with $\xi' \geq \eta'$ in $\mathcal{S}((\alpha,\beta+\gamma),a,b)$; and on the right by a point with $x = y$ formed similarly from the decreasing solution $(\xi,\eta)$ in $\mathcal{S}((\alpha+\beta,\gamma),a,b)$. If $M$ is smaller than the $x-$coordinate of the point in $\mathcal{S}$ satisfying $x \geq y = z$ and/or $m$ larger than the $z-$ coordinate of the point in $\mathcal{S}$ satisfying $x = y \geq z$ then upon watching the changes of $x-$ and $z-$coordinates of a point moving in the shadowed region, we see that the arc is defined on the left by a point of the form $(M,*,*)$ and/or on the the right by a point of the form $(*,*,m)$, and the claim follows again - the rhs-figure tries to illustrate these possibilities.

Let us now define the 'steepness' of a line through two points $Y' \in \{y=0\}$, $Z' \in \{z=0\}$ and as the ratio of the signed distances that $Y'$ and $X'$ have from $X$. So if $cx+dy+ez = f$ is *any* line, then lemma 3 yields the formula $\text{steepness}(l_f) = \frac{\gamma(\beta c-\alpha d)}{\beta(\gamma c-\alpha e)}$. Of course the steepness does not depend on $f$.

We now assume $c,d,e$ have the meaning and satisfy the hypotheses reserved for them. Define for real $f$ the line $l_f = \{cx+dy+ez = f\}$. The hypotheses ensure that the denominators figuring in the expressions given in lemma 3 for $\overrightarrow{XZ'}$ and $\overrightarrow{XY'}$, $Z' = Z'(f), Y' = Y'(f)$ - these being the points of intersections of the line $l_f$ with $z = 0$ and $y = 0$ respectively - are all positive. If $f = 0$ the fractions are indeed $> 1$. We see that as $f$ increases, $Y'$ and $Z'$ wander toward $X$. Let $\bar{l}$ be the line through points $(x_L,y_L,z_L)$ and $(x_R,y_R,z_R)$. It is a consequence of lemma 2a and the convexity of arc $\mathcal{S}'$ that we always shall have $0 < \text{steepness}(\bar{l}) < 1$. The minimizer of the function $(x,y,z) \mapsto cx+dy+ez$ restricted to $\mathcal{S}'$ is the subset of $\mathcal{S}'$ that line $l_f$ hits first as $f$ increases. This set consists of the point $(x_R,y_R,z_R)$ only if $\text{steepness}(l_f) > \text{steepness}(\bar{l})$, of both points $(x_L,y_L,z_L),(x_R,y_R,z_R)$ if the steepnesses are equal, and of point $(x_R,y_R,z_R)$ only if $\text{steepness}(l_f) < \text{steepness}(\bar{l})$. The line $\bar{l}$ obeys the equation $(y_R-y_L)x+(x_L-x_R)y+0z+(x_Ry_L-x_Ly_R) = 0$. Calculating its steepness, the claim follows. ∎

## 2. Maximizers and Minimizers

This section presents the main results; unfortunately many of these results are technical at first reading. However, just as is the case with lemma 5 of the previous section, large chunks of the technicalities evaporate for spaces in which $m = -\infty, M = +\infty$; so the reader is invited to always reflect first what happens in this case.

Fix a space $X = X(m,M,a,b;n;g)$ with $g$ as in proposition 1. It will be sometimes convenient to think of an $\mathbf{x} = (x_1,\ldots,x_n) \in X$ as augmented by $x_0 = M$, $x_{n+1} = m$ and to say that $\mathbf{x}$ has a *descent in* $X$ *at* $i \in \{0,1,\ldots,n\}$ if $x_i > x_{i+1}$. Sometimes we indicate the position of a descent at $i$ in a form like $(\ldots \overset{i}{>} \ldots)$, and write descents($\mathbf{x}$) for the set of descents of $\mathbf{x}$ (in X). We extend this definition in a natural manner to subtuples $\mathbf{x}(k:l)$. For example, if $\mathbf{x} = (3,3,1,1,-2) \in X(-5,3,6,*;5;g)$, then descents($\mathbf{x}$) $= \{2,4,5\}$, and descents($\mathbf{x}(3:4)$) is $\{0,2\}$ in terms of relative positions and $\{2,4\}$ in terms of absolute ones. No confusion will arise.

Let $\mathbf{x} \in X(m,M,*)$. If there exists an $\mathbf{i} = (i_\nu)_{\nu=0}^5 \in \mathbb{Z}^6$ satisfying $0 \leq i_0 < i_1 \leq i_2 < i_3 \leq i_4 < i_5 \leq n$, $r,s,t \in \mathbb{Z}_{>0}$, and reals $x,y,z$ such that $\mathbf{x}$ can be thought of as having the form

- $\mathbf{x} = \mathbf{x}(x,y,z) = (\ldots \overset{i_0}{>} x\mathbf{1}_r \overset{i_1}{\geq} \ldots \overset{i_2}{\geq} y\mathbf{1}_s \overset{i_3}{>} \ldots \overset{i_4}{>} z\mathbf{1}_t \overset{i_5}{\geq} \ldots)$, then $\mathbf{x}$ is amenable to an $\Uparrow\!\Downarrow\!\Uparrow - \mathbf{i}$-motion;

i.e. to a replacement $\mathbf{x} \mapsto \mathbf{x}' := \mathbf{x}(x',y',z')$ with $x' > x, y' < y, z' > z$ such that $\mathbf{x}' \in X$;

• $\mathbf{x} = \mathbf{x}(x,y,z) = (\ldots \overset{i_0}{\geq} x\mathbf{1}_r \overset{i_1}{>} \ldots \overset{i_2}{>} y\mathbf{1}_s \overset{i_3}{\geq} \ldots \overset{i_4}{\geq} z\mathbf{1}_t \overset{i_5}{>} \ldots)$, then $\mathbf{x}$ is amenable to an $\downarrow\!\uparrow\!\downarrow - \mathbf{i}$-motion; i.e. to a replacement $\mathbf{x} \mapsto \mathbf{x}' := \mathbf{x}(x',y',z')$ with $x' < x, y' > y, z' < z$ such that $\mathbf{x}' \in X$.

Note that $\uparrow\!\downarrow\!\uparrow - \mathbf{i}-$movability of $\mathbf{x}$ for some $\mathbf{i}$ is implied by (and implies) the existence of $i_0, i_3 \in$ descents$(\mathbf{x})$ such that $0 \leq i_0 < i_0 + 2 \leq i_3 \leq n-1$. In this case we can find $x,y,z,r,s,t,i_1,i_2,i_4,i_5$ (usually in many ways) such that $\mathbf{x}$ is $\uparrow\!\downarrow\!\uparrow - \mathbf{i}$-movable with $\mathbf{i} = (i_0,\ldots,i_5)$. Similarly there exist $i_1, i_5 \in$ descents$(\mathbf{x})$ with $1 \leq i_2 < i_2 + 2 \leq i_5 \leq n$ if and only if these can be completed to an $\mathbf{i} = (i_0,\ldots,i_5)$ (usually in various ways) such that $\mathbf{x}$ is $\downarrow\!\uparrow\!\downarrow - \mathbf{i}$-movable. A complete specification of a motion would make necessary also the indication of one of the amounts $x'-x, y'-y, z'-z$, but these quantities are irrelevant for us. Of course $\mathbf{i}$ and $r,s,t$ depend on each other; namely we have $i_1 - i_0 = r, i_3 - i_2 = s, i_5 - i_4 = t$. As with descents, movability depends via the values of $m$ and $M$ on the space $X$ to which we refer. Context will avoid confusions.

**Lemma 6** Given integers $k,l$ with $1 \leq k \leq l \leq n-1$, a space $X = X(m,M,a,b;n;g)$ and an $\mathbf{x} \in X$.
a. If there exist $i_0, i_3 \in$ descents$(\mathbf{x})$ with $0 \leq i_0 \leq k-1, l+1 \leq i_3 \leq n-1$, then $\mathbf{x} \notin$ maximizer$(S^{kl})$.
b. If there exist $i_0, i_3 \in$ descents$(\mathbf{x})$ with $0 \leq i_0 \leq k-2, k \leq i_3 \leq n-1$, then $\mathbf{x} \notin$ minimizer$(S^{kl})$.
b'. If there exist $i_0, i_3 \in$ descents$(\mathbf{x})$ with $0 \leq i_0 < i_0 + 2 \leq i_3 \leq l$, then $\mathbf{x} \notin$ minimizer$(S^{1l})$.
c. If there exist $i_2, i_5 \in$ descents$(\mathbf{x})$ with $1 \leq i_2 \leq l-1, l+1 \leq i_5 \leq n$, then $\mathbf{x} \notin$ maximizer$(S^{kl})$.
d. If there exist $i_2, i_5 \in$ descents$(\mathbf{x})$ with $1 \leq i_2 \leq k-2, l \leq i_5 \leq n$, then $\mathbf{x} \notin$ minimizer$(S^{kl})$.
d'. If there exist $i_2, i_5 \in$ descents$(\mathbf{x})$ with $l \leq i_2 < i_2 + 2 \leq i_5 \leq n$ then $\mathbf{x} \notin$ minimizer$(S^{1l})$.

Proof. a. Choose $i_0 \in$ descents$(\mathbf{x})$ maximal$\leq k-1$. Then (check that it is possible to) complete $i_0, i_3$ to an $\mathbf{i} = (i_0,\ldots,i_5)$ via choices $i_1 = k, s = 1, i_4 = i_3, t = 1$ such that $\mathbf{x}$ is $\uparrow\!\downarrow\!\uparrow - \mathbf{i}$-movable. Indeed, we can now think of $\mathbf{x}$ as having the form $\mathbf{x} = \mathbf{x}(x,y,z) = (\ldots \overset{i_0}{>} x\mathbf{1}_r \overset{k}{\geq} \ldots \geq y \overset{i_3}{>} z \geq \ldots)$. We subject $\mathbf{x}$ to an associated $\uparrow\!\downarrow\!\uparrow - \mathbf{i}-$motion $\mathbf{x} \mapsto \mathbf{x}' \in X$. Since $y$ and $z$ are to the right of the $l-$th entry, the motion increases exactly the $k-$th component of $\mathbf{x}$ by the amount $x' - x$, while leaving the other components of $\mathbf{x}(k:l)$ unaltered. Hence $S^{kl}(\mathbf{x}') > S^{kl}(\mathbf{x})$, proving the claim.

In the remainder of this proof we will be more succinct and specify $\mathbf{i}$ usually only to some necessary amount.

b. Choose $i_3$ minimal with $k \leq i_3$; $i_1 = i_2 = k-1$, $i_4 = i_3$. Then we can think of $\mathbf{x}$ as having the form $\mathbf{x} = (\ldots \overset{i_0}{>} x\mathbf{1}_r \overset{k-1}{\geq} y\mathbf{1}_s \overset{i_3}{>} z\mathbf{1}_t \overset{i_5}{\geq} \ldots)$. Consider an $\uparrow\!\downarrow\!\uparrow - \mathbf{i}-$motion $\mathbf{x} \mapsto \mathbf{x}'$. If $i_3 \geq l$ then all entries at positions in $\{k,\ldots,l\}$ decrease; if $i_3 < l$ then all entries changed at positions outside $\{k,\ldots,l\}$ increase. These positions define a nonempty set. Either way, the conservation of the sum of the entries of $\mathbf{x}$ by the motion (i.e. $S(\mathbf{x}) = S(\mathbf{x}')$) guarantees $S^{kl}(\mathbf{x}') < S^{kl}(\mathbf{x})$, proving the claim.

b'. Choose $i_3 \leq l$ maximal, put $i_4 = i_3$ and $t$ maximal. An $\uparrow\!\downarrow\!\uparrow - \mathbf{i}-$motion $\mathbf{x} \mapsto \mathbf{x}'$ replaces some entries of $\mathbf{x}$ outside positions $\{1,\ldots,l\}$ by larger values, while no such entry is replaced by a smaller value. So from sum conservation b' follows.

c. Choose $i_4 = i_3 = l$, $i_1 = i_2$ maximal$\leq l-1$, (i.e. $s$ maximal), $r = 1$, $i_5$ minimal with $l < i_5$ (i.e. $t$ maximal) Then we can think of $\mathbf{x}$ as $\mathbf{x} = (\ldots \overset{i_0}{\geq} x \overset{i_1}{>} y\mathbf{1}_s \overset{l}{\geq} z\mathbf{1}_t \overset{i_5}{>} \ldots)$. Subjecting $\mathbf{x}$ to a $\downarrow\!\uparrow\!\downarrow - \mathbf{i}-$motion to obtain $\mathbf{x}' \in X$ will in case $i_1 \leq k-1$ : increase all entries in $\mathbf{x}(k:l) = y\mathbf{1}_{s'}$ (for some $s' \leq s$), and in case $k-1 < i_1$ : decrease all entries moved outside of positions $\{k,\ldots,l\}$. Thus the sum-conservation implies $S^{kl}(\mathbf{x}') > S^{kl}(\mathbf{x})$.-

d. Choose $i_5$ minimal with $l \leq i_5$, and $i_2$ maximal$\leq k-2$, $i_1 = i_2$, $s = 1$, $i_3 = i_4$, $t$ maximal. We can think of $\mathbf{x}$ as $\mathbf{x} = (\ldots \overset{i_0}{\geq} x\mathbf{1}_r \overset{i_1}{>} y \overset{i_3}{\geq} z\mathbf{1}_t \overset{i_5}{>} \ldots)$. An associated $\downarrow\!\uparrow\!\downarrow-$motion $\mathbf{x} \mapsto \mathbf{x}'$ decreases all the entries in positions in $\{k,\ldots,l\}$ that are moved; hence $S^{kl}(\mathbf{x}) > S^{kl}(\mathbf{x}')$.-

d'. Choose $i_2$ minimal with the property $l \leq i_2$. Put $i_1 = i_2$ and $r$ maximal. We can now apply an $\downarrow\!\uparrow\!\downarrow - \mathbf{i}-$motion $\mathbf{x} \mapsto \mathbf{x}'$. All entries changed in positions $\{1,\ldots,l\}$ are diminished, so the claim follows. ∎

The following theorem answers completely which spaces $X$ are nonempty and, for certain types of points, whether they exist in a space $X$ or not. Such criteria can be useful in non-interactive software implementations of our method. A simple idea and results from [K] are used: the complicated formulae in (iv) arise from explicit enumeration of all vertices of a certain polytope given there; note that the third line of (iv) will be true only in a few exceptional cases. [K] works with ascending chains $m \leq x_1 \leq ... \leq x_n \leq M$ instead of the descending $M \geq x_1 \geq \ldots \geq m$ used here; to ease comparison, the notation follows that paper as closely as possible.

**Theorem 7** a. Let $X = X(m, M, a, b; n; g)$. If $m, M \in \mathbb{R}$, define $f(l) = (l-1)m + (n+1-l)M$. Then $X \neq \emptyset$ if and only if $ng(\frac{a}{n}) \leq b$, $m \leq M$, and one of the following conditions is satisfied, where quantifications are meant over integers:

i. $m = -\infty, M = +\infty$.

ii. $M \in \mathbb{R}$ and $(m = -\infty \vee a \in [f(2), f(1)])$ and $\exists \bar{n}, 1 \leq \bar{n} \leq n$, $(n-\bar{n})g(M) + \bar{n}g(\frac{a-(n-\bar{n})M}{\bar{n}}) \geq b$.

iii. $m \in \mathbb{R}$ and $(M = +\infty \vee a \in [f(n+1), f(n)])$ and $\exists \bar{n}, 1 \leq \bar{n} \leq n$, $\bar{n}g(\frac{a-(n-\bar{n})m}{\bar{n}}) + (n-\bar{n})g(m) \geq b$.

iv. $(m, M \in \mathbb{R}$ and $a \in ]f(n), f(2)[)$
$\wedge\ ((\exists n_1, n_2,\ 0 \leq n_1 < \frac{a-mn}{M-m},\ 0 \leq n_2 < \frac{Mn-a}{M-m},\ n_2 g(M) + (n-n_1-n_2)g(\frac{a-mn_1-Mn_2}{n-n_1-n_2}) + n_1 g(m) \geq b)$
$\vee(\ \exists n_1, n_2 \in \mathbb{Z}_{\geq 1},\ n_1 m + n_2 M = a,\ n_1 + n_2 = n$, and $n_2 g(M) + n_1 g(m) \geq b))$.

b. Given integers $n_1, n_2 \in \mathbb{Z}_{\geq 0}$, let $\tilde{a} = a - Mn_1 - mn_2$, $\tilde{b} = b - n_1 g(M) - n_2 g(m)$. Then $(M\mathbf{1}_{n_1}, \mathbf{x}, m\mathbf{1}_{n_2}) \leftrightarrow \mathbf{x}$ furnishes a 1-1 correspondence between the points of the form of the lhs($\leftrightarrow$) in $X(m, M, a, b; n; g)$ and the points in $\mathbf{x} \in X(m, M, \tilde{a}, \tilde{b}; n - n_1 - n_2; g)$. In particular this allows via (a) to decide whether $X(m, M, a, b; n; g)$ has points of the form $(M\mathbf{1}_{n_1}, \mathbf{x}, m\mathbf{1}_{n_2})$.

Proof. a. We apply proposition 1 with $k = n, \alpha_1 = \ldots = \alpha_n = 1$. Define $G(\mathbf{x}) = \sum_i g(x_i)$, $H = \{\mathbf{x} \in \mathbb{R}^n : \sum x_i = a\}$, $P = P(m, M, a; n) := \{\mathbf{x} \in H : M \geq x_1 \ldots \geq x_n \geq m\}, \mathcal{S} = \mathcal{S}(\mathbf{1}_n, a, b) = \{\mathbf{x} \in H : G(x) = b\}$, and $X = X(m, M, a, b; n; g)$. Clearly $X = \mathcal{S} \cap P$ and $P$ is convex. Let $\operatorname{ext} P$ be the extreme points of $P$.

Claim 1. $X \neq \emptyset \Leftrightarrow (G(\mathbf{m}) \leq b) \wedge (P$ is unbounded $\vee\ \exists \mathbf{p}\, \mathbf{p} \in \operatorname{ext} P,\ b \leq G(\mathbf{p}))$.

$\triangleright\Leftarrow$: The hypotheses imply of course that $P \neq \emptyset$; hence, as is easily seen, $\mathbf{m} = \frac{a}{n}\mathbf{1}_n \in P$. Now, there always exists a $\mathbf{p} \in P$ such that $b \leq G(\mathbf{p})$ : if $P$ is bounded this is part of the hypothesis, if $P$ is unbounded this follows by proposition 1 (or most directly by claim 4 in its proof). By convexity of $P$ and continuity of $G$, there exists a $\mathbf{p}' \in P$, $\mathbf{p}' \in [\mathbf{m}, \mathbf{p}] \subseteq H$ with $G(\mathbf{p}') = b$. So $\mathbf{p}' \in \mathcal{S} \cap P = X$.

$\Rightarrow$: The hypothesis implies of course $\mathcal{S} \neq \emptyset \neq P$, and so $\mathbf{m} \in P$ and by proposition 1, $G(\mathbf{m}) \leq b$. If $P$ is unbounded then rhs($\Leftrightarrow$) is trivially true. If $P$ is not unbounded, then $P$ is a convex polytope and hence the convex hull of its extreme points, called vertices. If we had $G(\mathbf{p}) < b$ for all $\mathbf{p} \in \operatorname{ext} P$, then Jensen's inequality [RV, p212c7] implies $G(\mathbf{x}) < b$ for all $\mathbf{x} \in P$, hence $\mathcal{S} \cap P = \emptyset$, a contradiction. $\triangleleft$

Note that $G(\mathbf{m}) = ng(\frac{a}{n})$, and that $P$ is unbounded iff $m = -\infty, M = +\infty$. It suffices, thus, to show the following claim.

Claim 2. ii $\vee$ iii $\vee$ iv $\Leftrightarrow (P$ is bounded $\wedge\ \exists \mathbf{p}\, \mathbf{p} \in \operatorname{ext}(P),\ b \leq G(\mathbf{p}))$.

$\triangleright\Rightarrow$: First, let $P$ satisfy ii. Then, since $M \in \mathbb{R}$, $P$ is bounded. If $a \in [f(2), f(1)]$, then $P(m, M) = P(-\infty, M)$, see [K, lemma 2f]. By [K, theorem 5], $\operatorname{ext}(P(-\infty, M))$ consists of the points $\mathbf{p} = \mathbf{p}(n_1, n_2) = (M\mathbf{1}_{n_2}, \xi\mathbf{1}_{n_1})$, $\quad n_1 \in \mathbb{Z}_{\geq 1},\ n_1 \leq n,\ n_1 + n_2 = n,\ \xi = \xi(n_1, n_2) = \frac{a-n_2 M}{n_1}$. Hence $G(\mathbf{p}(n_1, n_2)) \geq b$ is equivalent to $n_2 g(M) + n_1 g(\frac{a-n_2 M}{n_1}) \geq b$ for some $n_1$ and $n_2$. Thus rhs($\Leftrightarrow$) follows after a simple renotation. If $P$ satisfies iii, then one can infer rhs($\Leftrightarrow$) by using reasoning similar to that just given for ii, using [K, lemma 2f, theorem 6]; if $P$ satisfies iv, one uses [K, theorem 4d].

$\Leftarrow$: The hypothesis implies that $P$ is a nonempty polytope of one of the types $P(-\infty, M)$, $P(m, M)$ or $P(m, +\infty)$ with $m, M \in \mathbb{R}$. The polytopes $P(-\infty, M)$ and $P(m, +\infty)$ have known extreme points. Using them and the hypothesis yields ii and iii. Finally suppose that $P(m, M)$ satisfies the hypothesis. Since $P(m, M) \neq \emptyset$, it follows that $a \in [f(n+1), f(1)] = [f(n+1), f(n)] \cup ]f(n), f(2)[ \cup [f(2), f(1)]$ and these cases are precisely covered by ii,iii,iv and we are done. $\triangleleft$

b. The easy considerations leading to (b) are left to the reader. ∎

In theorems 8 an 9 below the structures of the sets maximizer$(S^{kl}|X)$ and minimizer$(S^{kl}|X)$, $1 \le k \le l \le n$, are determined. Note that theorem 8e relegates the calculation of the sets maximizer$(S^{kn})$, $2 \le k \le n$, to the calculation of the sets minimizer$(S^{1k-1})$. These latter are determined in theorem 9f. Theorem 9e relegates the calculation of sets minimizer$(S^{kn})$ for $2 \le k \le n$ to the calculation of the sets maximizer$(S^{1k-1})$. Note that the determination of the latter fall under the domain of theorem 8bcd and the answers are more explicit than those for the minimizers of the $S^{1k-1}$. Interestingly the optimizers of $S^{kl}$ depend usually only on one of the parameters $k, l$; hence by determining one of these optimizers we frequently have many others.

**Theorem 8** (maximizers) Given integers $k, l$ with $1 \le k \le l \le n-1$, and a space $X = X(m, M, a, b; n; g) \neq \emptyset$. Define the sets $U = \{\mathbf{x} \in X : \mathbf{x}(1:l) = M\mathbf{1}_l\}$, $V = \{(\xi\mathbf{1}_l, \eta\mathbf{1}_{n-l}) \in X : \xi, \eta \in \mathbb{R}\}$,
$W = \{\mathbf{x} \in X : \mathbf{x}(l+1:n) = m\mathbf{1}_{n-l}\}$. Then:
o. maximizer$(S^{1n}) = X$.
a. $\emptyset \neq$ maximizer$(S^{kl}) \subseteq U \cup V \cup W$; if two of the sets $U, V, W$ are nonempty, they are equal singletons.
b. If $U \neq \emptyset$, then maximizer$(S^{kl}) = U$.
c. If $V \neq \emptyset$, then there exist $\xi, \eta$, $M \ge \xi \ge \eta \ge m$ such that maximizer$(S^{kl}) = V = \{(\xi\mathbf{1}_l, \eta\mathbf{1}_{n-l})\}$.
d. If $W \neq \emptyset$, then, letting $\tilde{a} = a - (n-l)m$, $\tilde{b} = b - (n-l)g(m)$, and $\tilde{X} = X(m, M, \tilde{a}, \tilde{b}; l; g)$, there holds maximizer$(S^{kl}) = \{\mathbf{x} \in X : \mathbf{x}(1:k-1) \in$ minimizer$(S^{1k-1}|\tilde{X}), \mathbf{x}(l+1:n) = m\mathbf{1}_{n-l}\}$.
e. For $2 \le k \le n$, maximizer$(S^{kn})$ = minimizer$(S^{1k-1})$.

Proof. o: is obvious. a. Consider, to the extent existing, $\mathbf{u} \in U, \mathbf{v} \in V, \mathbf{w} \in W$. These $n$-tuples have the following forms: $\mathbf{u} = (M, \ldots, M, u_{l+1}, \ldots, u_n)$, $\mathbf{v} = (\xi, \ldots, \xi, \eta, \ldots, \eta)$, $\mathbf{w} = (w_1, \ldots, w_l, m, \ldots, m)$. From the facts that $S^{1n}(\mathbf{u}) = S^{1n}(\mathbf{v}) = S^{1n}(\mathbf{w}) = a$ and that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are decreasing with entries in $[m, M]$, it is easy to see that $\mathbf{u} \succeq \mathbf{v}$, $\mathbf{w} \succeq \mathbf{v}$, $\mathbf{u} \succeq \mathbf{w}$. Hence, if $\mathbf{u} \neq \mathbf{v}$, we have by [MO, p64c-4, p54c8] the contradiction $b = \sum g(u_i) > \sum g(v_i) = b$. Hypothesizing $\mathbf{u} \neq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$ leads to similar contradictions. Since $X$ is compact, and the functions $S^{kl}$ continuous, we know that maximizer$(S^{kl}) \neq \emptyset$. Now lemma 6ac tells us that if $\mathbf{x} \in$ maximizer$(S^{kl})$ then the following logical formula holds true: $((\text{descents}(\mathbf{x}) \cap \{0, \ldots, k-1\} = \emptyset) \vee (\text{descents}(\mathbf{x}) \cap \{l+1, \ldots, n-1\} = \emptyset)) \wedge ((\text{descents}(\mathbf{x}) \cap \{1, \ldots, l-1\} = \emptyset) \vee (\text{descents}(\mathbf{x}) \cap \{l+1, \ldots, n\} = \emptyset))$. Rewriting this formula as a disjunction of 4 conjunctions using the distributive law, the set of all $\mathbf{x}$ satisfying this formula is seen to be precisely $U \cup V \cup W$, yielding (a). b. The claim is obvious since for any $\mathbf{x} \in X \setminus U$ and $\mathbf{u} \in U$ we have of course $S^{kl}(\mathbf{x}) < (l-k+1)M = S^{kl}(\mathbf{u})$. c. For any real $\xi, \eta$ we have obviously $(\xi\mathbf{1}_l, \eta\mathbf{1}_{n-l}) \in X$ iff $(\xi, \eta) \in \mathcal{S}((l, n-l), a, b)$ and $M \ge \xi \ge \eta \ge m$. Now $\mathcal{S}((l, n-l), a, b)$ is by proposition 1 a point or a 'sphere' in the one-dimensional space $H_1 = \{(x, y) : lx + (n-l)y = a\}$; hence it consists of at most two points; $(\xi, \eta)$ and $(\xi', \eta')$ say which define a closed (possibly degenerated) interval containing $\frac{a}{n}\mathbf{1}_2$. Hence one of the points, $(\xi, \eta)$ say, satisfies $\xi \ge \eta$. d. Under the hypothesis of (d) we infer from (a) that maximizer$(S^{kl}) \subseteq W$, and hence $\mathbf{x}(l+1:n) = m\mathbf{1}_{n-l}$ for any $\mathbf{x} \in$ maximizer$(S^{kl})$. Fix such an $\mathbf{x}$. Then $S^{kl}(\mathbf{x}) \ge S^{kl}(\mathbf{x}')$ for all $\mathbf{x}' \in X$, hence $S^{1(k-1)}(\mathbf{x}) + S^{l+1,n}(\mathbf{x}) \le S^{1(k-1)}(\mathbf{x}') + S^{l+1,n}(\mathbf{x}')$ for all $\mathbf{x}' \in X$, in particular for those in $W$, for which latter we get $S^{1(k-1)}(\mathbf{x}) \le S^{1(k-1)}(\mathbf{x}')$. But $\mathbf{x}' \in W$ iff it is of the form $\mathbf{x}' = (\tilde{\mathbf{x}}, m\mathbf{1}_{n-l})$ with $\tilde{\mathbf{x}} \in \tilde{X}$. (d) follows. e. Is a consequence of $S^{1k-1}(\mathbf{x}) + S^{kn}(\mathbf{x}) = a$. ∎

**Theorem 9** (minimizers) Given integers $k, l$ with $2 \le k \le l \le n-1$, and a space $X = X(m, M, a, b; n; g) \neq \emptyset$. Define $U = \{\mathbf{x} \in X : \mathbf{x}(1:k-1) = M\mathbf{1}_{k-1}\}$, $V = \{(\xi\mathbf{1}_{k-1}, \eta\mathbf{1}_{n-k+1}) \in X : \xi, \eta \in \mathbb{R}\}$,
$W = \{\mathbf{x} \in X : \mathbf{x}(k:n) = m\mathbf{1}_{n-k+1}\}$. Then:
o. minimizer$(S^{1n}) = X$.
a. $\emptyset \neq$ minimizer$(S^{kl}) \subseteq U \cup V \cup W$; if two of the sets $U, V, W$ are nonempty, they are equal singletons.
b. If $U \neq \emptyset$, then letting $\tilde{a} = a - (k-1)M$, $\tilde{b} = b - (k-1)g(M)$, and $\tilde{X} = X(m, M, \tilde{a}, \tilde{b}; n-k+1; g)$, there holds minimizer$(S^{kl}) = \{\mathbf{x} \in X : \mathbf{x}(1:k-1) = M, \mathbf{x}(k:n) \in$ maximizer$(S^{l-k+2, n-k+1}|\tilde{X})\}$.
c. If $V \neq \emptyset$, then there exist $M \ge \xi \ge \eta \ge m$ such that minimizer$(S^{kl}) = V = \{(\xi\mathbf{1}_{k-1}, \eta\mathbf{1}_{n-k+1})\}$.

d. If $W \neq \emptyset$, then, minimizer$(S^{kl}) = W$.
e. For $2 \leq k \leq n$, minimizer$(S^{kn})$ = maximizer$(S^{1k-1})$.
f. The set minimizer$(S^{1l})$ can be obtained from an algorithm given below; it has only finitely many points each of which has the form $(M\mathbf{1}_r, x, y\mathbf{1}_s, z, m\mathbf{1}_t)$; if $(m, M) = (-\infty, +\infty)$, find the unique $(\xi, \eta) \in \mathcal{S}((n-1,1), a, b)$ with $\xi \geq \eta$, and the unique $(\xi', \eta') \in \mathcal{S}((1, n-1), a, b)$ with $\xi' \geq \eta'$, and calculate $\varepsilon = \text{sgn}(1 - l + \frac{\xi' - \xi}{\xi - \eta'})$.
Then

$$\text{minimizer}(S^{1l}) = \begin{cases} \{(\xi\mathbf{1}_{n-1}, \eta)\} & \text{if } \varepsilon = +1 \\ \{(\xi\mathbf{1}_{n-1}, \eta), (\xi', \eta'\mathbf{1}_{n-1})\} & \text{if } \varepsilon = 0 \\ \{(\xi', \eta'\mathbf{1}_{n-1})\} & \text{if } \varepsilon = -1. \end{cases}$$

Proof. o: is clear. a. The proof of (a) follows by consistently substituting in the proof of theorem 7a, $l$ by $k-1$, using that compactness of $X$ and continuity of $S^{kl}$ yields minimizer$(S^{kl}) \neq \emptyset$, and using lemma 6bd in a way similar to the usage of lemma 6ac in the proof of theorem 7a. b. The parts b,c,d, e are proved in complete analogy with the parts d, c, b, e respectively of theorem 7. The somewhat complicated indices appearing in $S^{\cdots}$ in (b) express in disguise the fact that one seeks to maximize the sum of the components $x_{l+1}, \ldots, x_n$. f. By lemma 6b'd', if $\mathbf{x} \in$ minimizer$(S^{1l})$, then there do not exist $i_0, i_3 \in$ descents$(\mathbf{x})$ with $0 \leq i_0 < i_0 + 2 \leq i_3 \leq l$ and there do not exist $i_2, i_5 \in$ descents$(\mathbf{x})$ with $l \leq i_2 < i_2 + 2 \leq i_5 \leq n$. In particular it follows that $|\text{descents}(\mathbf{x}(1:l))| \leq 2$ and $|\text{descents}(\mathbf{x}(l+1:n))| \leq 2$; where these descent sets have to be treated with $\mathbf{x}(1:l)$ considered as extended to $(x_0, \mathbf{x}(1:l), x_{l+1})$ and $\mathbf{x}(l+1:n)$ to $(x_l, \mathbf{x}(l+1:n), x_{n+1})$. Furthermore if the number of descents in one of these extended truncations of $\mathbf{x}$ is 2, then the descents are adjacent. It follows that the extended $\mathbf{x}(1:l)$ has one of the forms $(x_0 \geq x_1 \geq x_2 = \ldots = x_l = x_{l+1})$, $(x_0 = x_1 \geq x_2 \geq x_3 = \ldots = x_l = x_{l+1})$, ... $(x_0 = x_1 = x_2 = x_3 = \ldots \geq x_l \geq x_{l+1})$, where always one or both of the $\geq$ can be strict inequality signs. We can think of $(x_l, \mathbf{x}(l+1:n), x_{n+1})$ in a similar manner. It follows that every such $\mathbf{x}$ lies in one of the sets $U_{rst} := U_{r,s,t} := \{(M\mathbf{1}_r, x, y\mathbf{1}_s, z, m\mathbf{1}_t) \in X\}$, $r, s, t \in \mathbb{Z}_{\geq 0}, r + s + t = n - 2$. This gives us for the determination of minimizer$(S^{1l})$ the following algorithm:

- Determine the family $R = \{rst : U_{rst} \neq \emptyset\}$
- For each $rst \in R$ determine an $u_{rst} \in$ minimizer$(S^{1l}|U_{\mathbf{r}})$ as follows:
  Using in lemma 5 the definitions $\alpha = 1, \beta = s, \gamma = 1, c = 1, d = l - r - 1, e = 0$, $a(\text{L5}) = a - rM - tm$, $b(\text{L5}) = b - rg(M) - tg(m)$ (interpreting $0 \cdot \pm\infty = 0$), and consequently $\varepsilon = \text{sgn}\left(-l + r + 1 + \frac{x_L - x_R}{y_R - y_L}\right)$, determine $\{(\bar{x}, \bar{y}, \bar{z})\} :=$ minimizer$(1x + (l - r - 1)y|\mathcal{S}')$. Put $\mathbf{u}_{rst} = (M\mathbf{1}_r, \bar{x}, \bar{y}\mathbf{1}_s, \bar{z}, m\mathbf{1}_t)$.
- Calculate $m_{\mathbf{r}} = S^{1l}(u_{\mathbf{r}})$ and $m = \min\{m_{\mathbf{r}} : \mathbf{r} \in R\}$, and determine $R' = \{\mathbf{r} \in R : m_{\mathbf{r}} = m\}$.
- Then minimizer$(S^{1l}|X) = \bigcup_{\mathbf{r} \in R'}$ minimizer$(S^{1l}|U_{\mathbf{r}})$.

Finally, it is an easy task to see that in case $(m, M) = (-\infty, +\infty)$ the algorithm reduces to the process given in the announcement of theorem 9f. ∎

## 3. Infimum and Supremum of Spaces $X(*, a, *; n; *)$ in the Lattice $(\mathcal{D}(n, a), \preceq)$

Bapat [Ba, p62, 63] (and others) have observed that for $r = 0$ the set $\mathcal{D}(n, *) \cap \{\mathbf{x} : \mathbf{x} \geq r\}$, endowed with weak majorization order $\preceq_w$ has lattice theoretic properties. From this the corresponding properties follow at once for any fixed $r$. Note that: a subset of $\mathcal{D}(n, a)$ is bounded w.r.t. $\preceq$ $\Leftrightarrow$ it is bounded above w.r.t. $\preceq$ $\Leftrightarrow$ it is bounded in the sense of Euclidean metric. Furthermore every subset of $\mathcal{D}(n, a)$ is bounded below w.r.t. $\preceq$ by $\frac{a}{n}\mathbf{1}_n$.

We can now use part of Bapat's discussion to obtain part of the following (using partially his notations).

**Corollary 10** a. Let $X = X(m, M, a, b; n, g)$ be nonempty. For $l = 1, 2, \ldots, n$, let $\alpha_l = \inf_{\mathbf{x} \in X} S^{1l}(\mathbf{x})$ and $\alpha_l' = \sup_{\mathbf{x} \in X} S^{1l}(\mathbf{x})$. Define $\delta = \delta(X) = (\alpha_1, \alpha_2 - \alpha_1, \ldots, \alpha_n - \alpha_{n-1})$ and $\eta = \eta(X) = (\alpha_1', \alpha_2' -$

$\alpha'_1, \ldots, \alpha'_n - \alpha'_{n-1})$. Then $\delta$ and $\eta$ are infimum and supremum of $X$ in the partial order $(\mathcal{D}(n,a), \preceq)$; in other words $\delta$ and $\eta$ satisfy:
o. $\delta, \eta \in \mathcal{D}(n,a)$.
i. For all $\mathbf{x} \in X$, there holds $\delta \preceq \mathbf{x} \preceq \eta$.
ii. If $\delta', \eta'$ are any two elements for which the (corresponding) (o,i) hold, then $\delta' \preceq \delta \preceq \eta \preceq \eta'$.

b. $(\mathcal{D}(n,a), \preceq)$ is a conditionally complete lattice.

Proof. a. We noted that for all $\mathbf{x} \in X$ we have $\mathbf{x} \succeq \frac{a}{n}\mathbf{1}_n$ So with obvious modifications we can reason with our $X$ as Bapat [Ba, lemma 3+proof] does with (his) $S$, to obtain from $\sum_i \delta_i = \alpha_n = a$, the facts concerning $\delta$ claimed in (o), (i) and (ii). As for $\eta$, fix any $l \in \{2, \ldots, n-1\}$. Choose $\mathbf{v} \in$ maximizer$(S^{1,l-1})$, $\mathbf{v}' \in$ maximizer$(S^{1l})$, $\mathbf{v}'' \in$ maximizer$(S^{1,l+1})$. By theorem 8a, $\mathbf{v}'$ is in one of the sets $U, V, W$ constructed for the pair $(k,l) = (1,l)$ and it follows that $(\mathbf{v}', \mathbf{v}')_\downarrow \in \tilde{X} = X(m, M, 2a, 2b; 2n, g)$ is in the corresponding $\tilde{U}, \tilde{V}$ or $\tilde{W}$ defined in the space $\tilde{X}$ for the pair $(1, 2l)$. ¿From this it follows that $(\mathbf{v}', \mathbf{v}')_\downarrow \in$ maximizer$(S^{1,2l}|\tilde{X})$ and so $2\alpha'_l = S^{1,2l}((\mathbf{v}', \mathbf{v}')_\downarrow) \geq S^{1,2l}((\mathbf{v}, \mathbf{v}'')_\downarrow) \geq \alpha'_{l-1} + \alpha'_{l+1}$. This inequality is also seen to hold for $l = 1$ if we put $\alpha'_0 := 0$. Thus $\eta$ is decreasing. Since $S^{1l}(\eta) = \alpha'_l$, it follows that $\mathbf{x} \preceq \eta$ for all $\mathbf{x} \in X$; the remaining thing to show for $\eta$ is also an immediate consequence of its construction.

b. We have to show that every nonempty subset $S$ of $\mathcal{D}(n,a)$, bounded below/above w.r.t. $\preceq$, has an infimum/supremum (in $\mathcal{D}(n,a)$). We have observed in (a) that for showing the existence of an infimum one can proceed precisely as in Bapat adding the fact that the elements of $S$ all have sum $a$, so as to permit expressing his claims with $\preceq$ instead of with $\preceq_w$. Bapat's corollary 4 can be used in a similar fashion to show the existence of a supremum for a set $S$ bounded above. ∎

It is interesting to note that while Bapat's non-constructive supremum definition $\eta_{\mathrm{Bap}}$ is applicable to our spaces $X$ and we have by his uniqueness proofs $\eta_{\mathrm{Bap}}(X) = \eta(X)$, the two $\eta$s are not always the same. E.g. if we apply our construction to the set $S = \{(12,2,2,2), (6,6,6,0)\}$, then we obtain the non-decreasing element $\eta = (12,2,4,0)$; the trade-off for our constructiveness is, hence, less general validity.

## 4. Applications to Eigenvalue Estimations

In this section we show how our results can be applied to estimate eigenvalues, given the traces of certain powers and/or the determinant of a matrix having only real eigenvalues. The functions $\mathbb{R} \ni x \mapsto x^{2k}$, $\mathbb{R}_{\geq 0} \ni x \mapsto x^{2k+1}$, and $\mathbb{R}_{>0} \ni x \mapsto -\ln(x)$ are strictly convex. The idea is to use that for an $n \times n-$matrix $A$ and any $k \in \mathbb{Z}_{>0}$, we have $\mathrm{trace}(A^k) = \sum_{i=1}^n \lambda_i^k$, and, if $A$ has positive eigenvalues, $-\ln(\det(A)) = -\sum \ln \lambda_i$.

To gauge our estimates, which are throughout best possible (apart of rounding which was done at the second decimal mostly) against an explicit example the reader may wish to use the real symmetric matrix

$$\tilde{A} = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix} \text{ that has } \left\{ \begin{array}{rcl} \lambda & = & (9.376, 6.423, 4.775, 1.426), \\ \mathrm{trace}\tilde{A} & = & 22, \\ \mathrm{trace}\tilde{A}^2 & = & 154, \\ \mathrm{trace}\tilde{A}^3 & = & 1201, \\ \mathrm{trace}\tilde{A}^4 & = & 9954, \\ \det \tilde{A} & = & 410. \end{array} \right. ,$$

and which has appeared in [WS].

As follows from the remarks preceding theorem 8, once determined the optimizers of the $S^{ll}$, we actually know most of the optimizers of the $S^{kl}$, and therefore the best possible estimates for sums of eigenvalues of the type $\lambda_k + \ldots + \lambda_l$. So we limit ourselves to just indicate the bounds for the eigenvalue themselves.

Problem 1. Let $\mathcal{A}$ be the class of all $4\times4$ matrices $A$ of real eigenvalues with trace 22 (i.e. trace$A = 22$) and the trace of whose square is 154 (i.e. trace$A^2 = 154$.) Give bounds for the eigenvalues.

Solution: The 4−tuple of eigenvalues $\lambda(A) = (\lambda_1 \geq \ldots \geq \lambda_4)$ of $A \in \mathcal{A}$ lies in the space $X = X(-\infty, +\infty, 22, 154; 4; x^2)$. We know beforehand that $\lambda(\tilde{A}) \in X$ but could use theorem 7a to infer from $4(\frac{22}{4})^2 = 121 < 154$ that $X \neq \emptyset$. For each $A \in \mathcal{A}$ we have $S^{ll}(\text{minimizer}(S^{ll})) \leq \lambda_l(A) \leq S^{ll}(\text{maximizer}(S^{ll}))$. Since $m = -\infty, M = +\infty$, the sets $U$ and $W$ of theorems 8 and 9 are empty for all $l \in \{1, 2, 3\}$ for which they are defined; the sets $V$ are consequently nonempty and obtainable from the decreasing solutions $(\xi, \eta)$ to the systems $\mathcal{S}((l, 4-l), 22, 154)$. For $l = 1, 2, 3$ one obtains the solutions $\mathbf{v}_1 = (10.48 \cdot \mathbf{1}_1, 3.84 \cdot \mathbf{1}_3)$, $\mathbf{v}_2 = (8.37 \cdot \mathbf{1}_2, 2.63 \cdot \mathbf{1}_2)$, $\mathbf{v}_3 = (7.16 \cdot \mathbf{1}_3, 0.53 \cdot \mathbf{1}_1)$, respectively. Using theorems 9c and 8c for $k = l = 2$ we obtain $S^{22}(\mathbf{v}_1) = 3.84 \leq \lambda_2(A) \leq S^{22}(\mathbf{v}_2) = 8.37$, similarly for $k = l = 3$ we obtain $S^{33}(\mathbf{v}_2) = 2.63 \leq \lambda_3(A) \leq S^{33}(\mathbf{v}_3) = 7.16$ Using theorem 9e for $k = n = 4$, and then theorem 8c, we have $S^{44}(\text{minimizer}(S^{44})) = S^{44}(\text{maximizer}(S^{13})) = S^{44}(\mathbf{v}_3) = 0.53 \leq \lambda_4(A)$, while $\lambda_4(A) \leq S^{44}(\text{maximizer}(S^{44})) = S^{44}(\text{minimizer}(S^{13}))$. Now minimizer$(S^{13})$ is determined via the algorithm given in the proof of theorem 9f with $l = 3$. In that algorithm of course $r = t = 0, s = n - 2 = 2$. One finds minimizer$(S^{13}) = \mathbf{v}_3$; and hence the upper bound $\lambda_4(A) \leq 3.84$.

Problem 2. Let $\mathcal{A}$ be the class of all $4 \times 4$ matrices $A$ of real positive eigenvalues with trace$A = 22$, and determinant $\det(A) = 410$. Give bounds for the eigenvalues.

Solution: $\ln(\det(A)) = 6.02$. Similarly as in Problem 1, we have to search for minimizer$(S^{ll})$ and maximizer$(S^{ll})$; this time in space $X(-\infty, +\infty, 22, -6.02; 4; g)$ defined by the proper strictly convex function $g(x) = \begin{cases} -\ln x & \text{for } x > 0 \\ +\infty & \text{for } x \leq 0 \end{cases}$. The union of the spaces $V$ is given this time by the 4-tuples $\mathbf{v}_1 = (12.36 \cdot \mathbf{1}_1, 3.21 \cdot \mathbf{1}_3), \mathbf{v}_2 = (8.66 \cdot \mathbf{1}_2, 2.34 \cdot \mathbf{1}_2), \mathbf{v}_3 = (6.92 \cdot \mathbf{1}_3, 1.24 \cdot \mathbf{1}_1)$, from where we read off the bounds $\lambda_1(A) \leq 12.36$, $S^{22}(\mathbf{v}_1) = 3.21 \leq \lambda_2(A) \leq S^{22}(\mathbf{v}_2) = 8.66$, $S^{33}(\mathbf{v}_2) = 2.34 \leq \lambda_3(A) \leq S^{33}(\mathbf{v}_3) = 6.92$, $1.24 \leq \lambda_4(A)$. Invoking theorem 9f, we also find minimizer$(S^{11}) = \{\mathbf{v}_3\}$ and maximizer$(S^{44}) = \{\mathbf{v}_1\}$, thus we have the bounds $6.92 \leq \lambda_1(A)$ and $\lambda_4(A) \leq 3.21$.

Note that a matrix with real eigenvalues $\lambda_1 = \lambda_2 = 12.606$, and $\lambda_3 = \lambda_4 = -1.606$ and has the specified trace and determinant, but does not satisfy the bounds given. Reason: the negative eigenvalues impede rewriting the information on the determinant as a sum of logarithms of eigenvalues equality.

Another reason for requiring nonnegative eigenvalues happens when admission of negativity leads us out of the convexity region of the function $g$ as is the case with $g(x) = x^3$.

Problem 3. Let $\mathcal{A}$ be the class of all $4 \times 4$ matrices $A$ of real nonnegative eigenvalues with trace$A = 22$ and trace$A^3 = 1201$. Give bounds for the eigenvalues.

Solution: We apply our theorems for the space $X = X(0, +\infty, 22, 1201; 4; x^3)$. It happens that there exist 4-tuples $(\xi \mathbf{1}_l, \eta \mathbf{1}_{4-l}) \in X$ for $l = 1, 2$. Indeed minimizer$(S^{22}) = \{(10.04 \cdot \mathbf{1}_1, 3.99 \cdot \mathbf{1}_3)\} =$ maximizer$(S^{11})$ and minimizer$(S^{33}) = \{(8.35 \cdot \mathbf{1}_2, 2.65 \cdot \mathbf{1}_2)\} =$ maximizer$(S^{22})$. So the estimates easiest to obtain are $\lambda_1(A) \leq 10.04$, $3.99 \leq \lambda_2(A) \leq 8.35$, $2.65 \leq \lambda_3(A)$. Furthermore minimizer$(S^{44}) =$ maximizer$(S^{13}) = \{\mathbf{x} \in X : x_4 = 0\}$ by theorems 9e and 8ad, for it happens so that the (only) point of form $(\xi \cdot \mathbf{1}_3, \eta \cdot \mathbf{1}_1)$ in $X(-\infty, +\infty)$, namely $(7.37, -.11)$ does not lie in $X(0, +\infty)$. So we find $0 \leq \lambda_4$. To find a lower bound for $\lambda_1(A)$ we invoke theorem 9f with $l = 1$. The algorithm yields $r = 0, s = 1, t = 1$, hence minimizer$(S^{11}) = \{(7.71 \cdot \mathbf{1}_2, 6.59 \cdot \mathbf{1}_1, 0)\}$, and so $7.71 \leq \lambda_1(A)$. Finally we find an upper bound for $\lambda_4(A)$. By theorem 8e, maximizer$(S^{44}) =$ minimizer$(S^{13})$. To find this minimizer, theorem 9f tells us to put $l = 3, r = 0$. With $t = 0, s = 2$, we find minimizer$(S^{13}) = \{(10.04 \cdot \mathbf{1}_1, 3.99 \cdot \mathbf{1}_3)\}$, and so $\lambda_4(A) \leq 3.99$.

Problem 4. Let $\mathcal{A}$ be the class of all $4 \times 4$ matrices $A$ of real eigenvalues with trace$A = 22$ and trace$A^4 = 9954$. Give bounds for the eigenvalues.

Solution: We can use the space $X(-\infty, +\infty, 22, 9954; 4; x^4)$. One then finds by the techniques of the previous problems without difficulties that minimizer($S^{22}$) = maximizer($S^{11}$) = maximizer($S^{44}$) = minimizer($S^{13}$) = $\{(9.77{\cdot}\mathbf{1}_1, 4.08{\cdot}\mathbf{1}_3)\}$, minimizer($S^{33}$) = $\{(8.38{\cdot}\mathbf{1}_2, 2.62{\cdot}\mathbf{1}_2)\}$ = maximizer($S^{22}$), and minimizer($S^{11}$) = minimizer($S^{44}$) = $\{(7.59{\cdot}\mathbf{1}_3, -.77{\cdot}\mathbf{1}_1)\}$ = maximizer($S^{33}$). Thus we have $7.59 \le \lambda_1(A) \le 9.77$, $4.08 \le \lambda_2(A) \le 8.38$, $2.62 \le \lambda_3(A) \le 7.59$, $-0.77 \le \lambda_4(A) \le 4.08$.

## 5. Notes and Concluding Remarks

a. There should be no difficulties in extending our theorems to the case of continuous not necessarily differentiable convex functions by using the theory of subgradients and subdifferentials [R, p214c-2, p215c2]. In particular a combination of [R, p223c5] and the fact [R, p264c2] should permit to establish proposition 1, claim 3. The essence of lemma 2 is that if we move via $s \mapsto (x(s), y(s), z(s))$ along the curve $\mathcal{S}$, then $(x, y, z)$ change according to $\upuparrows\!\downarrow\!\uparrow$ or $\downdownarrows\!\uparrow\!\downarrow$. This can be established via the theory of majorization. Indeed on the plane $x + y + z = a$ such changes to an $(x', y', z')$ are the only possible that allow escaping majorization-comparability (and hence by [MO, p92c8] inequalities of the type $g(x) + g(y) + g(z) < g(x') + g(y') + g(z')$ or its reverse).

b. To what degree could one of the defining conditions of our spaces $X$, namely $(*)$ $\sum_i g(x_i) = b$ with a strictly convex function $g$ be substituted by a more general one? Few of the more conventional generalizations seem to be possible without at least some technical troubles. Do our theorems remain essentially valid if one substitutes $g$ by an unbounded quasiconvex function, i.e. a function with convex level-sets? If not before, we expect major troubles in lemma 2b. While an unbounded convex function, even after 'tilting' it with respect to its defining domain yields convex levelsets, a quasiconvex function does not have this property. We also expect troubles with the validity of lemma 2b if we generalize $(*)$ to $\sum_i g_i(x_i) = b$ with $g_i$ convex. The reason is that results of the type $\mathbf{x} \preceq \mathbf{y} \Rightarrow \sum_i \alpha_i g_i(x_i) \le \sum_i \alpha_i g_i(y_i)$, do *not* hold in general (but are valid and were used if all $g_i$ are equal to a certain convex $g$, see [MO, p92c-3]).

c. Could we generalize, on the spaces defined in this paper, and with its methods, the functions $S^{kl}$ to be optimized? Again not much seems to be possible in this direction. We run into troubles in lemma 6. Let $\dot{S}^{kl} : \mathbb{R}^n \to \mathbb{R}$ be a function dependent only of variables $x_k, \ldots, x_l$. At the bottom, in lemma 6 we invariably use the reasoning that a positive or negative change of the value $\sum_{i=k}^{l} x_i$, that is of $S^{kl}(\mathbf{x})$, be reflected in a positive or negative change of the value $\dot{S}^{kl}(\mathbf{x})$. This means that we have $S^{kl}(\mathbf{x}) = S^{kl}(\mathbf{y})$ iff $\dot{S}^{kl}(\mathbf{x}) = \dot{S}^{kl}(\mathbf{y})$. Hence the function $\dot{S}^{kl}(\mathbf{x}) = g(S^{kl}(\mathbf{x}))$ for some monotone realvalued function $g$. So the generalizations possible in the sense intended are trivial.

d. The observation (c) is deplorable, for it means for example that we cannot deduce from our results that, given a space $X = X(0, +\infty, a, b; n; x^2)$, one has for certain $\xi, \eta, \xi', \eta'$ that minimizer($\prod_{i=1}^n x_i$) = $\{(\xi\mathbf{1}_1, \eta\mathbf{1}_{n-1})\}$ and maximizer($\prod_{i=1}^n x_i$) = $\{(\xi\mathbf{1}_{n-1}, \eta\mathbf{1}_1)\}$; a fact equivalent to one proven in [BSW].

e. Given a decreasing $n$-tuple $\mathbf{a}$, put $\tilde{X} = X(-\infty, +\infty, a, b; n; x^2) \cap \{\mathbf{x} : \mathbf{a} \succeq \mathbf{x}\}$. Using Kuhn-Tucker theory, [GJSW, p308] find that the structure of maximizer($S^{11}|\tilde{X}$) is of the form $(\xi\mathbf{1}_1, \eta\mathbf{1}_t, \mathbf{a}(t{+}1 : n))$. Can problems of this type solved by our method?

f. Summing up, in a sense we now understand better why we so often find a simple structure for the optimizers. And yet, as the results mentioned in (d) and (e) painfully remind us, our understanding is still incomplete.

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