Maximum K_{r+1}-free graphs which are not r-partite

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Abstract

Turán's theorem states that the maximum size of a K_{r+1}-free graph G of order n is attained by a complete r-partite graph. Here we determine the maximum size of G on the additional restriction that G is not r-partite. Also, we present a new proof of the result of Andrásfai, Erdős, and Gallai on the maximum size of an order-n graph whose shortest odd cycle has given length $2l+1$. The extremal graphs are characterized for all feasible values of parameters.

1 Introduction

Let K_m denote the complete graph on m vertices. The fundamental theorem of Turán [6] states that $\mathrm{ex}(n, K_{r+1})$, the maximum size of a K_{r+1}-free graph G of order n, is attained by the complete r-partite graph with parts whose sizes differ by at most 1. This result is one of the cornerstones of extremal graph theory. For future use, let the *Turán graph* $T_r(n)$ be the corresponding extremal graph and let

$$t_r(n) = e(T_r(n)) = \mathrm{ex}(n, K_{r+1}).$$

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It is interesting to know what happens if we additionally require that G is not r-partite, that is, we consider the graphs in

$$\mathcal{G}_{n,r} = \{G : v(G) = n,\ G \not\supset K_{r+1},\ \chi(G) > r\}.$$

The classical paper of Andrásfai, Erdős and Sós [1] determines $\max\{\delta(G) : G \in \mathcal{G}_{n,r}\}$, the largest minimum degree of $G \in \mathcal{G}_{n,r}$. Erdős and Simonovits [4] studied the more general problem of maximizing $\delta(G)$ over

$$\mathcal{H}_{n,r,s} = \{G : v(G) = n,\ G \not\supset K_{r+1},\ \chi(G) > s\}.$$

(Note that $\mathcal{G}_{n,r} = \mathcal{H}_{n,r,r}$.)

In this paper we investigate

$$p_r(n) = \max\{e(G) : G \in \mathcal{G}_{n,r}\}.$$

This problem is completely settled in Section 2. Namely, we will prove the following result.

Theorem 1 *Let $n \ge r+3$ and $r \ge 2$. If $r > \frac{n-1}{2}$, then $p_r(n) = t_r(n) - 2$. If $r \le \frac{n-1}{2}$, then*

$$p_r(n) = t_r(n) - \left\lfloor \frac{n}{r} \right\rfloor + 1. \tag{1}$$

Moreover, the extremal graphs are characterized by Theorem 4 and Lemma 5.

For convenience, the bound (1) on $p_r(n)$ is stated relative to the Turán function $t_r(n) = \mathrm{ex}(n, K_{r+1})$ so that it is immediately obvious how many edges we lose by imposing the restriction $\chi(G) > r$. Note also that $\mathcal{G}_{n,r} = \emptyset$ for $n \le r+2$ or for $r = 1$. (In order to see the former claim, note that the complement of any K_{r+1}-free graph G of order $r+2$ must contain either two disjoint edges or a triangle, so $\chi(G) \le r$.)

It would be nice to compute $\max\{e(G) : G \in \mathcal{H}_{n,r,s}\}$. It is easy to see that this function equals $t_r(n) - O(n)$ if r, s are fixed. However, the exact computation seems to be a hard task in general.

For $r = 2$, we have canonical examples of graphs of chromatic number larger than r: odd cycles. Häggkvist [5] studied the problem which odd cycles must be present in a non-bipartite graph of given minimum degree. Andrásfai and, independently, Erdős and Gallai (see Erdős [3, Lemma 1]) computed $b_l(n)$, the maximum size of a non-bipartite graph of order n whose shortest odd cycle has length $2l+1$. Unfortunately, this result is not well known, appearing only as an auxiliary lemma in [3]. In fact, we were unaware of it until we rather accidentally came across [3] while revising the current paper. Since no attempt to characterize the extremal graphs was made in [3] and our proof is different (albeit longer), we decided to keep it. Here is the precise statement of the result.

Theorem 2 *Let $l \geq 2$ and $n \geq 2l+1$. Then*

$$b_l(n) = \left\lfloor \frac{(n-2l+3)^2}{4} \right\rfloor + 2l - 3\,. \tag{2}$$

All extremal graphs are described by the construction at the beginning of Section 3.

One of the extremal graphs for the $b_l(n)$-problem is obtained by taking C_{2l+1} and appropriately cloning two adjacent vertices, but there are other constructions.

It is easy to see that the right-hand side of (2) strictly decreasing as a function of l for fixed n, $2 \leq l \leq \frac{n-1}{2}$. Hence, $b_l(n)$ is also equal to the maximum size of a non-bipartite graph G of order n without any odd cycle of length less than $2l+1$.

Thus the problems of determining $p_r(n)$ and $b_l(n)$ overlap in a special case: $b_2(n) = p_2(n)$. A more remarkable relation is that our proofs of Theorems 1 and 2 are based on the same idea of Erdős [2].

2 Determining $p_r(n)$

A construction, giving a lower bound on $p_r(n)$, can be obtained as follows. Let $n \geq r+3$. Choose integers

$$1 \leq n_1 \leq \cdots \leq n_r \text{ such that } \sum_{i=1}^{r} n_i = n-1 \text{ and } n_{r-1} \geq 2, \tag{3}$$

and pairwise disjoint sets $N_1, \ldots, N_r$, where $|N_i| = n_i$. Let s and t be the two smallest indices i (in either order) for which $n_i > 1$. (Thus $|s-t| = 1$.) Let $S = [r] \setminus \{s,t\}$. Choose any subset $A \subset N_s$ which is *proper* (that is, $A \neq \emptyset$ and $A \neq N_s$). Choose $y \in N_t$. To $K_r(N_1, \ldots, N_r)$, the complete r-partite graph on $N_1 \cup \cdots \cup N_r$, add a vertex x connected to everything in $(\cup_{i\in S} N_i) \cup (\{y\} \cup A)$ but remove all edges between y and A. Let us call the obtained graph $G = G(\mathbf{n})$, where $\mathbf{n} = (n_1, \ldots, n_r)$. Of course, the isomorphism class of G depends on the choice of $|A|$ (and the choice of s,t if $n_s \neq n_t$) but the size $e(G)$ does not:

$$e(G) = \sigma_2(\mathbf{n}) + \sigma_1(\mathbf{n}) - n_s - n_t + 1, \tag{4}$$

where $\sigma_2(\mathbf{n}) = \sum_{i<j} n_i n_j$ and $\sigma_1(\mathbf{n}) = \sum_i n_i$. More generally, for an arbitrary (not necessarily increasing) sequence $\mathbf{n}$ with at least two entries larger than 1, let $G(\mathbf{n})$ be obtained by properly ordering $\mathbf{n}$ and then taking the above construction.

It is easy to see that $K_{r+1} \not\subset G$. Indeed, if some $(r+1)$-set $K \subset V(G)$ spanned a complete graph, then $x, y \in K$ (because $G - x$ and $G - y$ are r-partite); however, $\Gamma(x) \cap \Gamma(y) = \cup_{i\in S} N_i$ is $(r-2)$-partite, a contradiction. (Here, $\Gamma(x)$ denotes the neighborhood of x.)

Also, we have $\chi(G) > r$. Indeed, suppose on the contrary that we can color G with r colors. Choosing arbitrary $y_s \in N_s \setminus A$, $y_t \in N_t$, and $y_i \in N_i$, $i \in S$, we obtain a copy of K_r, so the colors of these vertices do not depend on their choices. But then x and each vertex of A see the same set of $r-1$ colors among its neighbors. Hence the set $A \cup \{x\}$ is monochromatic, which is a contradiction as x is connected to A.

Let us turn to proving upper bounds on $p_r(n)$. As we have already observed, $\mathcal{G}_{n,r} = \emptyset$ for $n \le r+2$. Therefore we restrict our attention to $n \ge r+3$. First we prove the required upper bound in the following special case.

Lemma 3 *Let $r \ge 2$ and $n \ge r+3$. Let $G \in \mathcal{G}_{n,r}$ be such that for some vertex y we have $\chi(G-y) = r$. Then $e(G) \le e(G(\mathbf{n}))$ for some $\mathbf{n}$ satisfying (3).*

Proof. Take an r-coloring of $G-y$. Let $\{y_1\}, \dots, \{y_l\}, N_1, \dots, N_{r-l}$ be the color classes, of which l have size one. Let $n_1, \dots, n_{r-l}$, all at least 2, be the sizes of $N_1, \dots, N_{r-l}$, respectively. As $n \ge r+3$, we have $l < r$.

Let $Y = \{y, y_1, \dots, y_l\}$. Note that Y spans the complete subgraph for otherwise G is r-colorable. Let

$$M_i = \{x \in N_i : \Gamma(x) \supset Y\}$$

and $m_i = |M_i|$. By reordering, let us assume that $m_1 \le \dots \le m_{r-l}$.

We claim that each M_i is non-empty. Otherwise, for every $x \in N_i$ choose $f(x) \in Y$ such that $\{x, f(x)\} \notin E(G)$. The $l+1$ sets $\{z\} \cup \{x \in N_i : f(x) = z\}$, $z \in Y$, are independent and partition $Y \cup N_i$. Together with the $r-l-1$ remaining parts N_j, $j \ne i$, this gives an r-coloring of G, contradicting our assumption.

Thus each $m_i \ge 1$. Moreover, $l \le r-2$ for otherwise $Y \cup \{x\}$ for some $x \in M_1$ spans a copy of K_{r+1}.

Let $\bar{e}_{ij}$ be the number of edges missing between N_i and N_j. Potentially, Y creates $\prod_{i=1}^{r-l} m_i$ copies of K_{r+1}. A missing edge between N_i and N_j destroys at most $\frac{1}{m_i m_j} \prod_{h=1}^{r-l} m_h$ such copies. Hence,

$$\prod_{h=1}^{r-l} m_h \le \sum_{i<j} \frac{\bar{e}_{ij}}{m_i m_j} \prod_{h=1}^{r-l} m_h \le \frac{\sum_{i<j} \bar{e}_{ij}}{m_1 m_2} \prod_{h=1}^{r-l} m_h. \tag{5}$$

We see that $\sum_{i<j} \bar{e}_{ij}$, the total number of edges missing between the N_i's, is at least $m_1 m_2$. Thus

$$e(G) \le \sigma_2(1^{(l)}, n_1, \dots, n_{r-l}) - m_1 m_2 + l + \sum_{i=1}^{r-l} m_i, \tag{6}$$

where $1^{(l)}$ means the number 1 repeated l times. A simple optimization shows that it is best to take $m_i = n_i$ for $i \ge 3$, and $m_1 = 1$ (recall $m_1 \le m_2$), which gives us $e(G) \le e(G(1^{(l)}, n_1, \dots, n_{r-l}))$, as required. ∎

Now we are ready to prove our main result whose proof relies on Lemma 3.

Theorem 4 *Let $r \geq 2$ and $n \geq r+3$. Then $p_r(n)$ equals the maximum of $e(G(\mathbf{n}))$ over all integers satisfying (3). Moreover, all extremal graphs are described by our construction.*

Proof. Our argument is built upon the ideas from Erdős' proof [2] of Turán's theorem, where it is shown that the degree sequence of a K_{r+1}-free graph can be majorized by that of an r-partite graph.

Let $G \in \mathcal{G}_{n,r}$ have the maximum size. We prove the theorem by induction on r. We do not give a separate proof for the base case $r = 2$: the inductive step, when specialized to $r = 2$, gives a self-contained proof. We prove the desired bound first and then analyze the cases of equality.

Let $V = V(G)$. Choose $x \in V$ with its degree being equal to the maximal degree of G, that is, $d(x) = \Delta(G)$. Let $D = \Gamma(x)$. Let the graph H be obtained from G by removing all edges inside $C = V \setminus D$ and adding all edges between D and C.

We claim that $H \not\supset K_{r+1}$. Indeed, suppose otherwise. The vertex set K of this K_{r+1} must intersect C because $G[D]$ and $H[D]$ are the same. As C is an independent set in H, we have $|K \cap C| = 1$. By the symmetry of C, we can assume that $K \cap C = \{x\}$. But then K spans a complete graph in G, a contradiction.

Note that for every $y \in V$ we have $d_G(y) \leq d_H(y)$: if $y \in C$, this follows from $d_G(y) \leq \Delta(G) = d_H(y)$; if $y \in D$, then $\Gamma_G(y) \subset \Gamma_H(y)$.

If $H[D]$ is not $(r-1)$-partite, then by the induction assumption we have $e(H[D]) \leq e(G(\mathbf{n}))$ for some $\mathbf{n} = (n_1, \ldots, n_{r-1})$. Let the r-vector $\mathbf{m}$ be obtained from $\mathbf{n}$ by inserting the number $|C|$. We have

$$e(G) \leq e(H) \leq (n - |C|)\,|C| + e(G(\mathbf{n})) \leq e(G(\mathbf{m})),$$

proving the required upper bound.

Hence, we can assume that $H[D]$ is $(r-1)$-partite: $D = \cup_{i=1}^{r-1} D_i$. (For $r = 2$ we get this conclusion for free: $D = \Gamma(x)$ is an independent set because $G \not\supset K_3$.) Let $d_i = |D_i|$, $d = |D|$, and $c = |C|$. We can assume that each d_i is at least 2 for otherwise the required upper bound follows by Lemma 3. Also, $c \geq 2$ for otherwise $G = H$ is r-partite.

Call a part D_i *good* if there is $y_i \in D_i$ which is connected in G to everything in $V \setminus D_i$. We claim that all, but at most one, parts are good. (We assume here that $r \geq 3$ as the claim is vacuously true for $r = 2$.) Suppose on the contrary that, for example, D_1 and D_2 are bad. Let the r-vector $\mathbf{d}$ be made of the numbers $d_1, \ldots, d_{r-1}, c$. We have

$$e(G) \leq \sigma_2(\mathbf{d}) - \frac{d_1 + d_2}{2},$$

which strictly beats the desired bound. Indeed, assuming $d_1 \leq d_2$, we have

$$\sigma_2(\mathbf{d}) - \frac{d_1 + d_2}{2} \leq \sigma_2(\mathbf{d}) - d_1 \leq e(G(d_1, d_2 - 1, d_3, \ldots, d_{r-1}, c)) - 1.$$

Note that the last sequence has at least two elements which are at least 2 (namely, d_1 and c), so it still satisfies (3). This upper bound on $e(G)$ contradicts the maximality of G.

We also obtain a contradiction by assuming that all parts are good: if $G[C]$ is empty, then $\chi(G) \le r$; otherwise $G \supset K_{r+1}$. So, let D_1 be the unique bad part. If each vertex of D_1 misses at least 2 neighbors in C, then

$$e(G) \le \sigma_2(\mathbf{d}) - 2d_1 < \sigma_2(\mathbf{d}) - d_1,$$

which is too small. Hence, there is $y_1 \in D_1$ such that $C \setminus \Gamma(y_1)$ consists of a single vertex z. Choose $y_i \in D_i$, $i \in [2, r-1]$, which witnesses the fact that D_i is good. Then $\{y_1, \dots, y_{r-1}\}$ is an $(r-1)$-clique which is connected (in G) to everything in $C \setminus \{z\}$. Hence, this set is independent, which implies that $\chi(G - z) \le r$. Now, we can apply Lemma 3 again. The upper bound is proved.

Let us characterize the cases of equality. We go over the proof of the upper bound, using the same notation.

If $G[D]$ is not $(r-1)$-partite, then by induction $G[D] \cong G(\mathbf{m})$ for some $\mathbf{m}$. Moreover, each vertex $y \in D$ is connected in G to everything in C: otherwise $d_G(y) < d_H(y)$ and $e(G) < e(H)$, a contradiction to the maximality of G. It follows that $G[C]$ is the empty graph and G is as desired.

Suppose that $G[D]$ is $(r-1)$-partite. Our proof shows that there is a vertex y, either $y \in C$ or $y \in D$, such that $G - y$ is $(r-1)$-partite. Let the parts of $V' = V \setminus \{y\}$ be $\{y_1\}, \dots, \{y_l\}, N_1, \dots, N_{r-l}$. (Now we use the notation of Lemma 3.) Of all possible choices of y and an $(r-1)$-partition of V', take one which minimizes l.

We must have $m_i = n_i$ for $i \ge 3$ and $m_1 = 1$. As we have equality in (5), all missing edges in $G[N_i, N_j]$ lie between M_1 and $M_2 \cup \dots \cup M_k$, where $m_2 = \dots = m_k$. In fact, all missing edges lie inside just *one* $G[M_1, M_i]$ for otherwise starting with $Y \cup M_1$ we can greedily add $z_i \in M_i$, consecutively for $i = 2, \dots, r-l$, to get a K_{r+1}-subgraph.

The case $m_2 = n_2$ is impossible: otherwise we can move the vertex in M_1 into N_2 to obtain another legitimate $(r-1)$-partition of $V \setminus \{y\}$ with the new M_1 being empty, which is a contradiction as we already know.

If some y_i is not connected to some $u \in V \setminus Y$, then u belongs to N_j with $n_j = 2$. (Otherwise, moving u to the part $\{y_i\}$ we obtain a new legitimate r-partition of $V \setminus \{y\}$ with a smaller l.) Then either $j = 1$ or $j = 2$ (because $M_i = N_i$ for $i \ge 3$). As we have equality in (6), every vertex of $V \setminus Y$ has at most one non-neighbor in Y.

It is impossible that some two vertices $w, z \in Y$ have degree less than $n-1$ each. Otherwise, it follows from the above information that $n_1 = n_2 = 2$, the graph $G[N_1 \cup N_2 \cup \{w, z\}]$ contains a perfect matching in its complement and so is 3-colorable, which implies that $\chi(G) \le r$, a contradiction.

Thus, all missing edges in $G[Y, V \setminus Y]$ are between $(N_1 \setminus M_1) \cup (N_2 \setminus M_2)$ and some $z \in Y$, which is exactly what we want. Now we know all the edges of G and in order to maximize $e(G)$ we must have that n_1, n_2 are the two smallest numbers among the n_i's, which is precisely what our construction says.

The theorem is proved. ∎

It remains to describe which sequences $\mathbf{n}$ give us $p_r(n)$.

Lemma 5 *Let $n \geq r+3 \geq 5$. If $r > \frac{n-1}{2}$, then $\mathbf{n} = (1^{(2r-n+1)}, 2^{(n-r-1)})$ is the (unique) sequence satisfying (3) and maximizing (4). If $r \leq \frac{n-1}{2}$, then the optimal sequences are precisely those $\mathbf{n}$ in (3) which satisfy all of the following inequalities:*

$$
\begin{aligned}
n_1 &\geq 2, && (7)\\
n_2 &\leq n_1 + 1, && (8)\\
n_r &\leq n_1 + 2, && (9)\\
n_r &\leq n_3 + 1. && (10)
\end{aligned}
$$

Proof. Let $\mathbf{n}$ be an optimal sequence and $s = \min\{i \in [r] : n_i \geq 2\}$. If $s \geq 2$, then decreasing n_r by 1 and increasing n_{s-1} by 1, we increase $e(G(\mathbf{n}))$ by $n_r + n_{s+1} - 4 \geq n_r - 2$. By the maximality of $\mathbf{n}$ we conclude that $n_r = 2$, $r > \frac{n-1}{2}$, and $\mathbf{n}$ is precisely the required (unique) sequence.

So, suppose that $n_1 \geq 2$ and thus $r \leq \frac{n-1}{2}$. If we move 1 from n_r to n_1, then $e(G(\mathbf{n}))$ increases by at least $n_r - n_1 - 2$, implying (9). The remaining inequalities are proved likewise. We have shown that any optimal sequence has the stated form.

Finally, let us consider, for $r \leq \frac{n-1}{2}$, the set $\mathcal{N}$ of sequences satisfying (3) and (7)–(10). It routine to see that $1 \leq |\mathcal{N}| \leq 3$ and $e(G(\mathbf{n}))$ is constant on $\mathcal{N}$. It follows that $\mathcal{N}$ is precisely the set of optimal sequences. ∎

Remark. Note that, depending on n and r, there are from one to three different sequences satisfying (3) and (7)–(10).

Proof of Theorem 1. Let $r \leq \frac{n-1}{2}$, for example. It is easy to see that among all sequences $\mathbf{n}$ satisfying (3) and (7)–(10), we can choose one satisfying $n_r \leq n_1 + 1$. This choice of parts is the same as the choice for $T_r(n-1)$. Now, to construct the Turán graph $T_r(n)$ we add one more vertex to the smallest part, that is, $n_2 + \cdots + n_r$ extra edges. Comparing this with the construction of $G(\mathbf{n})$ and using the fact that $n_2 = \lfloor n/r \rfloor$, we obtain (1). ∎

3 Determining $b_l(n)$

Recall that $b_l(n)$ is the maximum size of a graph of order n containing a cycle of length $2l+1$ but no odd cycle of strictly smaller length. Trivially, $b_1(n) = \binom{n}{2}$ so let us assume that $l \ge 2$.

A lower bound on $b_l(n)$ is given by the following construction. Take the Turán graph $T_2(n-2l+3)$; let its parts be X and Y, $|X|-|Y| \in \{-1,0,1\}$. Let $x \in X$ and let A be an arbitrary proper subset of Y, that is, $A \ne \emptyset$ and $A \ne Y$. Add a set L of $2l-3$ vertices spanning a path; let its end-vertices be u and v. Remove all edges between x and A but add edges $\{x,u\}$ and $\{v,y\}$ for all $y \in A$.

As A is a proper subset of Y, the constructed graph G contains a $(2l+1)$-cycle. On the other hand, the removal of any vertex of L makes G bipartite. Hence, any odd cycle C must traverse all vertices of L. But we need at least 4 more vertices in order to connect u to v. This implies that C has at least $2l+1$ vertices, as required.

It is easy to see that the size of G is given by the formula (2) of Theorem 2. We claim that this construction is optimal.

Proof of Theorem 2. We have to prove the upper bound on $b_l(n)$. For $l=2$, the theorem follows from the results of Section 2 because all $p_2(n)$-extremal graphs contain a 5-cycle. So, let us assume that $l \ge 3$. If $n = 2l+1$, then $b_l(n) = n$ with C_{2l+1} being the only available graph. So, assume that $n > 2l+1$.

Let G be a graph attaining $b_l(n)$. Let $V = V(G)$.

Let $x \in V$ be a vertex of maximum degree and $Y = \Gamma(x)$ be its neighborhood. We have $|Y| \ge 3$. Let y be a vertex of Y of the largest degree; let $X = \Gamma(y)$. As $K_3 \not\subset G$, the sets X and Y are independent and $X \cap Y = \emptyset$. Let $C \subset G$ be a $(2l+1)$-cycle visiting vertices $v_1, v_2, \ldots, v_{2l+1}$ in this order. Define

$$\begin{aligned} L &= V(C) \setminus (X \cup Y), \\ F &= G[X \cup Y], \\ R &= V \setminus (X \cup Y \cup L), \\ r &= |R|, \\ H &= G[\,\overline{R}\,] \;=\; G[X \cup Y \cup L]. \end{aligned}$$

We claim that $|L| \ge 2l-3$. Let us suppose the contrary. As F is bipartite, we can find distinct $a, b \in X \cup Y$ which are connected by a path P whose interior points lie inside L such that the length of P is odd if and only if a, b lie in the same part. This gives us the required contradiction. For example, if a, b lie in different parts, then we can connect them by a path $P' \subset F$ of length one or three (use x, y if $\{a,b\} \notin E(G)$). But then $P \cup P'$ is an odd cycle of length at most $(2l-4)+3$, a contradiction.

Suppose first that $|L| = 2l - 3$, that is, C intersects $X \cup Y$ in 4 vertices. The properties of $x, y \in F$ imply, after a moment's thought, that C has precisely two vertices in each part of F and, moreover, these 4 vertices are consecutive vertices of C, say, $v_1, v_3 \in X$ and $v_2, v_4 \in Y$.

Clearly, C is an induced cycle for otherwise we can find a shorter odd cycle. Also, it is routine to see that the vertices $v_6, \ldots, v_{2l}$ of C cannot send any edges to $X \cup Y$ for otherwise we can find a shorter odd cycle by using the vertices x, y. Likewise, $\Gamma(v_5) \cap X = \Gamma(v_{2l+1}) \cap Y = \emptyset$ for otherwise we can find a 5-cycle, contradicting our assumption $l \geq 3$. Let $A = \Gamma(v_{2l+1}) \cap X \neq \emptyset$, $a = |A|$, $B = \Gamma(v_5) \cap Y \neq \emptyset$, and $b = |B|$.

We already know the H-degrees of all vertices in L: $(b + 1, 2, 2, \ldots, 2, a + 1)$. Please also notice that we cannot have any edges between A and B for it would create a shorter odd cycle. By the choice of x, the H-degree of each vertex in $X \setminus A$ (resp. A) is at most $|Y|$ (resp. $|Y| - b + 1$). Likewise, each H-degree in $Y \setminus B$ (resp. B) is at most $|X|$ (resp. $|X| - a + 1$). And, of course, every vertex has degree at most $\Delta(G) = |Y|$ in G.

This shows that

$$\begin{aligned} 2e(G) &\leq 2e(H) + 2\sum_{z \in R} d_G(z) \;\leq\; (b+1) + (a+1) + (|L| - 2) \times 2 \\ &+ (|X| - a)|Y| + a(|Y| - b + 1) + (|Y| - b)|X| + b(|X| - a + 1) + 2r\,|Y| \\ &= 2\big((|X| + r) \cdot |Y| + a + b - ab + |L| - 1\big). \end{aligned}$$

A simple optimization, in view of $|X| + |Y| + r + |L| = n$ and the inequalities $a \geq 1$ and $b \geq 1$, shows that we have to take $\min(a, b) = 1$ and that the sets Y and $X \cup R$ must be nearly equal. This gives

$$e(G) \leq \left\lfloor \frac{(n - |L|)^2}{4} \right\rfloor + |L|. \tag{11}$$

Recall that $|L| = 2l - 3$, obtaining precisely the required upper bound

In order to characterize the extremal graphs (at least in this case) let us show that $R = \emptyset$. Suppose on the contrary that R is non-empty. Then there are no edges connecting R to $(X \setminus A) \cup (Y \setminus B)$ because each such edge is counted three times in the bound on $2e(G)$. Similarly, R spans no edge in G but every $z \in R$ has degree $\Delta(G)$. As $\min(a, b) = 1$, it follows that $\max(a, b) = |Y| - 1$ and each vertex $z \in R$ is connected to everything in $A \cup B$. But then $G \supset C_5$, a contradiction. Now, it easily follows from $R = \emptyset$ that G is given by our construction.

Hence, let us assume that for any choice of a $(2l+1)$-cycle C we have $|L| \geq 2l - 2$. Suppose first that there is C intersecting both X and Y. Because the diameter of the bipartite graph F is 3, we can assume by symmetry that $v_1 \in X$ and then that $v_2 \in Y$ or $v_4 \in Y$. (We do not exclude the possibility that $|V(C) \cap (X \cup Y)| = 3$.)

Let us consider the case $v_2 \in Y$. By threatening to create a $(2l + 1)$-cycle having 4 vertices in $X \cup Y$ or a shorter odd cycle via x, y, one can show that there is no edge between $X \cup Y$ and $\{v_4, \ldots, v_{2l}\}$. Thus v_3, v_{2l+1} are the only vertices of L that are connected to $X \cup Y$. Repeating the argument leading to (11), but with respect to $X' = X \cup \{v_3\}$, $Y' = Y \cup \{v_{2l+1}\}$, and $L' = L \setminus \{v_{2l+1}, v_1, v_2, v_3\}$, we obtain (11) (with L' instead of L). However, this time the inequality is strict, because when we passed from X, Y to X', Y' at least one of these sets strictly increased its size. This is the desired contradiction.

If $v_4 \in Y$, then one can argue that v_2, v_3 are the only vertices of L that can be connected to $(X \cup Y) \setminus \{v_1, v_4\}$ and the same contradictory bound arises.

The case when C intersects, say X, in two vertices u, v reduces to the above case: replace the even uv-path along C by two edges $\{u, y\}$ and $\{y, v\}$, obtaining an odd cycle of length at most $2l + 1$ which intersects both X and Y.

Therefore, it remains to consider only the case when every $(2l+1)$-cycle C intersects $X \cup Y$ in at most one vertex. Suppose first that $v_1 \in X$. Then v_2, v_{2l+1} cannot be connected to Y. (Otherwise we get K_3 or C_5.) Likewise, v_3, v_{2l} cannot be connected to X. Moreover, $\Gamma(v_3) \cap \Gamma(v_{2l}) = \emptyset$. Also, none of $v_4, \ldots, v_{2l-1}$ can be connected to $X \cup Y$ for otherwise we would get a shorter odd cycle or a cycle intersecting both X and Y, a contradiction. Thus

$$\sum_{v \in L} d_H(v) \leq 2(|X| + 1) + (|Y| + 4) + (2l - 4) \times 2 = 2|X| + |Y| + 4l - 2.$$

Similarly to what we did before (and using $|X| = n - |Y| - 2l - r$), we obtain

$$\begin{aligned} 2e(G) &\leq 2|X| \cdot |Y| + \big(2|X| + |Y| + 4l - 2\big) + 2r|Y| \\ &= |Y| \big(2n - 2|Y| - 4l - 1\big) + 2n - 2r - 2. \end{aligned}$$

The optimal choice here is $r = 0$ and $y = \lfloor \frac{n}{2} \rfloor - l$. By considering odd/even $n \geq |L| + 4 = 2l + 4$ separately, one can see that the obtained bound is strictly smaller than (2), a contradiction. Similarly, we show that $v_1 \in Y$ is also impossible.

Finally, assume that every $(2l + 1)$-cycle is disjoint from $X \cup Y$. Suppose first that v_1 is connected to, for example, X. The routine analysis shows that, in order to prevent a shorter odd cycle or a C_{2l+1} intersecting $X \cup Y$, there are no edges between the following pairs of sets: $(X, \{v_2, v_4, v_{2l-1}, v_{2l+1}\})$, $(Y, \{v_1, v_3, v_{2l}\})$ and $(X \cup Y, \{v_5, \ldots, v_{2l-2}\})$. Moreover, the X-neighborhoods of v_1, v_3, v_{2l} are disjoint as well as the Y-neighborhoods of $v_2, v_4, v_{2l-1}, v_{2l+1}$. We conclude that

$$\begin{aligned} 2e(G) &\leq 2|X| \cdot |Y| + |X| + |Y| + (2l + 1) \times 2 + 2r|Y| \\ &= 2\,|Y| \big(n - |Y| - 2l - 1\big) + n - r + 2l + 1. \end{aligned} \tag{12}$$

Optimizing, we take $|Y| = \lfloor \frac{n-1}{2} \rfloor - l$ and $r = 0$. One can see that the obtained bound is strictly less than (2) for $n \geq |L| + 4 \geq 2l + 5$.

If there are no edges between C and $X \cup Y$, then we get a strictly better bound than (12).

This finishes the proof of the theorem. ∎

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