Game Chromatic Index of Graphs with Given Restrictions on Degrees

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Abstract

Given a graph G and an integer k, two players alternatively color the edges of G using k colors so that adjacent edges get different colors. The *game chromatic index* $\chi'_g(G)$ is the minimum k for which the first player has a strategy that ensures that all edges of G get colored.

The trivial bounds are $\Delta(G) \le \chi'_g(G) \le 2\Delta(G) - 1$, where $\Delta(G)$ denotes the maximal degree of G. Lam, Shiu, and Xu and, independently, Bartnicki and Grytczuk asked whether there is a constant C such that $\chi'_g(G) \le \Delta(G) + C$ for every graph G. We show that the answer is in the negative by constructing graphs G such that $\chi'_g(G) \ge 1.008\,\Delta(G)$ and $\Delta(G) \to \infty$. On the other hand, we show that for every $\mu > 0$ there is $\varepsilon > 0$ such that for any graph G with $\Delta(G) \ge (1/2+\mu)v(G)$, we have $\chi'_g(G) \le (2-\varepsilon)\Delta(G)$, where $v(G)$ denotes the number of vertices of G.

Keywords: Edge coloring of graphs, game chromatic index, games on graphs, random strategies
[*]Research supported in part by NSF grant DMS-0701183
[†]Research supported in part by NSF grant CCR-0502793
[‡]Research supported in part by NSF grant DMS-0457512

1 Introduction

Let a graph G and a positive integer k be given. Two players, called Alice and Bob, alternatively color a previously uncolored edge of G in one of the colors from $[k] = \{1, \ldots, k\}$ so that no two adjacent edges have the same color. Thus, at any moment of the game, the current partial coloring of $E(G)$ is a proper edge coloring. The game can end in two different ways. Either all edges of G are colored (and then Alice is the winner) or the uncolored edge picked by a player cannot be properly colored (and then Bob wins).

Let us agree that Alice starts the game. (In fact, all theorems stated in this paper will remain valid for the version where we let Bob start the game.) The *game chromatic index* $\chi'_g(G)$ is the smallest k such that Alice has a winning strategy. This parameter has been previously studied by Lam, Shiu and Xu [9], Cai and Zhu [6], Erdős, Faigle, Hochstättler, and Kern [8], Andres [1], Bartnicki and Grytczuk [2], and others.

This is a variation of the *game chromatic number* which is analogously defined for the game where nodes (not edges) are colored. The latter parameter is much better studied; we refer the reader to Bohman, Frieze, and Sudakov [5] for some history and references on the game chromatic number.

The trivial bounds on the game chromatic index are

$$\Delta(G) \leq \chi'_g(G) \leq 2\Delta(G) - 1, \tag{1}$$

where $\Delta(G)$ denotes the maximal degree of G.

Unfortunately, the game chromatic index seems hard to analyze. For example, a player's move can easily harm that player later in the game. Also, it is not clear if there is any useful 'potential' function that measures a player's progress. Therefore, we settle for the modest task of getting a constant factor improvement over the trivial bounds (1) when $\Delta(G)$ is large.

Lam, Shiu and Xu [9, Question 1] and, independently, Bartnicki and Grytczuk [2, Problem 1] asked whether there is a constant C such that $\chi'_g(G) \leq \Delta(G) + C$ for an arbitrary graph G. In Section 2 we show that the answer to this question is in the negative. Namely, we construct, for every sufficiently large d, a graph G with $\Delta(G) \leq d$ and $\chi'_g(G) \geq 1.008\, d$.

On the other hand, the lower bound in (1) is attainable for some graphs. A trivial example is $G = K_{1,d}$. However, we believe that large minimal degree $\delta(G)$ will force $\chi'_g(G)$ to be well above $\delta(G)$. Namely, we make the following conjecture.

Conjecture 1 *There are $\varepsilon > 0$ and d_0 such that any graph G with $\delta(G) \geq d_0$ satisfies $\chi'_g(G) \geq (1+\varepsilon)\delta(G)$.*

Of course, the conclusion of Conjecture 1 is interesting only when $\Delta(G) < (1+\varepsilon)\delta(G)$, that is, when all degrees are fairly close to each other.

From the other direction, we show in Section 3 that for any $\mu > 0$ there is $\varepsilon > 0$ such that any graph G with $\Delta(G) \geq (1/2 + \mu)v(G)$ satisfies $\chi'_g(G) < (2 - \varepsilon)\Delta(G)$. (Here, $v(G)$ denotes the number of vertices of G.) Surprisingly, this is done by letting Alice play randomly, see Section 3 for details. While probabilistic intuition and reasoning often help in the analysis of combinatorial games, see e.g. Beck [3], there are not many examples where non-trivial results are obtained by actually introducing randomness into a player's strategy. Such examples were discovered by Spencer [11], Bednarska and Łuczak [4], Pluhár [10], and others. Our proof of the upper bound fits into this category.

The restriction $\Delta(G) \geq (1/2 + \mu)v(G)$ in the above result is needed in order to make our proof work. We do not believe that there is anything special about the constant 1/2 here. We conjecture that a much stronger claim is true.

Conjecture 2 *There is $\varepsilon > 0$ such that for an arbitrary graph G we have $\chi'_g(G) \leq (2 - \varepsilon)\Delta(G)$.*

2 Lower Bounds

Theorem 3 *For every sufficiently large integer d, there is a graph G with maximum degree at most d such that $\chi'_g(G) > 1.008\, d$.*

Proof. Let $\beta = 4/7$, $\alpha = 1 - \beta = 3/7$, and $\lambda = 1/25$. Let d be sufficiently large and let $n = \lfloor d/\beta - 3d^{2/3} \rfloor$.

We define a graph G of order n and maximum degree at most d as follows. Take two disjoint sets A and B of sizes $\lceil \alpha n \rceil$ and $\lfloor \beta n \rfloor = n - \lceil \alpha n \rceil$ respectively. The vertex set of G is $A \cup B$. Put a complete bipartite graph between A and B. Let A be an independent set. Let $G[B]$, the subgraph of G induced by B, be a random graph with each pair of B being an edge with probability $p = 1 - \alpha/\beta = 1/4$, independently of the other pairs.

Let the acronym *whp* (with high probability) mean 'with probability $1 - o(1)$ as $n \to \infty$'.

By the Chernoff bound [7], whp the G-degree $d(x)$ of every $x \in V(G)$, satisfies $|d(x) - \beta n| \le n^{2/3}$. In particular, the maximal degree of G is at most d. Also, whp every subset X of B spans at least $p\binom{|X|}{2} - n^{5/3}$ edges. (Indeed, any fixed X violates this inequality with probability $o(2^{-n})$ by the Chernoff bound.) Fix any $G[B]$ that satisfies these two conditions.

Let $k \le 1.008d$ be an arbitrary integer and let $\kappa = k/n$. In order to prove Theorem 3 we have to show that Bob has a winning strategy for the pair (G, k). By (1), it is enough to consider only those k that are at least $\Delta(G)$.

At the start of the game, Bob picks some $l = \lceil \lambda n \rceil$ *special* colors, say $1, \dots, l \in [k]$. (Note that the assumption $k \ge \Delta(G)$ implies that $l \le k$.) His strategy consists of two stages. At each round of *Stage 1*, Bob tries to color some (arbitrary) edge inside B with one of the special colors. If this is impossible (that is, the endpoints of every uncolored edge of $G[B]$ see all special colors between the two of them), then Stage 1 is over. In *Stage 2* Bob plays arbitrarily.

Let us show that Bob necessarily wins. Suppose on the contrary that all edges of G get colored by the end of the game.

Let Stage 1 have τn^2 rounds. Suppose that Alice plays $\tau_B n^2$ times inside B (and $(\tau - \tau_B)n^2$ times between A and B).

Let us analyze the moment when Stage 1 ends. Take any special color $i \in [l]$. Let $X_i \subseteq A \cup B$ be the set of vertices that are adjacent to an edge of color i. Let $\mu_i = |B \setminus X_i|/n$. We have

$$p\binom{\mu_i n}{2} - n^{5/3} = \left(\frac{p\mu_i^2}{2} + o(1)\right) n^2 \le \tau_B n^2 + \frac{l\mu_i n}{2} + o(n^2). \tag{2}$$

Indeed, $G[B \setminus X_i]$ has at least $p\binom{\mu_i n}{2} - n^{5/3}$ edges and all of them must be colored. On the other hand, Bob can color at most $l\mu_i n/2$ edges of $G[B \setminus X_i]$ because he uses only the l special colors in Stage 1 (and each color class is a matching).

Inequality (2), which is quadratic in μ_i, implies that

$$\mu_i \le \frac{\lambda + \sqrt{\lambda^2 + 8p\tau_B}}{2p} + o(1). \tag{3}$$

Also, we have

$$2\tau_B n^2 \le \sum_{i=1}^{l} \left(|X_i \cap B| - 3|X_i \cap A|\right). \tag{4}$$

Indeed, Bob increases $\sigma = \sum_{i=1}^{l}(|X_i \cap B| - 3|X_i \cap A|)$ by 2 in each his move in Stage 1. If Alice uses a special color on a $G[A, B]$-edge, then σ does not change in this round. If Alice does something else, which happens at least $\tau_B n^2$ times, then σ increases by at least 2 during the round.

For $i \in [l]$, let Y_i be the set of vertices covered by the edges of Color i after the game ends. Let $\mu n^2 = \sum_{i=1}^{l} \mu_i n$. At most μn^2 edges can be colored with special colors in Stage 2, because an edge colored $i \in [l]$ in Stage 2 must intersect $B \setminus X_i$. (Recall that A is an independent set in G.) Thus we have

$$\sum_{i=1}^{l}(|Y_i \cap B| - 3|Y_i \cap A|) \geq \sum_{i=1}^{l}(|X_i \cap B| - 3|X_i \cap A|) - 2\mu n^2. \tag{5}$$

The total number of edges of special colors at the end of the game is, by (4) and (5),

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{l}|Y_i| &= \frac{1}{2}\sum_{i=1}^{l}(|Y_i \cap A| + |Y_i \cap B|) \\ &\leq \frac{2}{3}\sum_{i=1}^{l}|Y_i \cap B| - \frac{\tau_B - \mu}{3}n^2 \leq \frac{2\lambda\beta}{3}n^2 - \frac{\tau_B - \mu}{3}n^2 + o(n^2). \end{aligned}$$

Each of the remaining $k - l \geq (\kappa - \lambda)n$ colors is used on at most $n/2$ edges. Since the total number of edges of G is $(\beta/2 + o(1))n^2$, we have

$$\frac{2\lambda\beta}{3} - \frac{\tau_B - \mu}{3} + \frac{\kappa - \lambda}{2} \geq \frac{\beta}{2} + o(1).$$

By dividing by $\beta/2$, re-arranging, and using the definition of μ and Inequality (3), we obtain

$$\frac{\kappa}{\beta} + o(1) \geq 1 + \frac{2\tau_B}{3\beta} - \frac{4\lambda}{3} - \frac{\lambda^2 + \lambda\sqrt{\lambda^2 + 8p\tau_B}}{3p\beta} + \frac{\lambda}{\beta}. \tag{6}$$

By taking the derivative with respect to τ_B of the right-hand side of (6), one can conclude that the minimum over all real τ_B is attained when $\tau_B = 3\lambda^2/(8p)$. Substituting this into (6) and using the known values of the constants, we obtain that

$$\frac{\kappa}{\beta} + o(1) \geq \frac{3781}{3750} > 1.0082. \tag{7}$$

This contradiction shows that Bob wins, finishing the proof of Theorem 3. ∎

Remark. In order to have a rigorous proof of Theorem 3 checkable by hand, we used rational numbers for all fixed constants. These choices are not optimal (given the stated inequalities) but are good rational approximations of such. In particular, the bound (7) can be slightly improved. Further improvements can be obtained by using more sophisticated strategies for Bob in Stage 1. Unfortunately, the analysis becomes too messy while the new bounds seem still to be very close to 1. Therefore we settled for the current version.

3 Upper Bounds

Here we are going to prove the upper bound on χ'_g promised in the Introduction. Our result will be stronger if we give Bob the freedom to skip moves. Namely, we consider the following new game, studied by Andres [1].

Let G and k be given. Bob and Alice alternatively make moves. Bob starts. In his move, Bob can either properly color an uncolored edge or skip (that is, not color any edge at all). Alice, however, always has to properly color an uncolored edge. As in the old version, any moment of the game gives a partial proper coloring of $E(G)$ and Alice wins if the whole graph is colored at the end. Let the *upper game chromatic index* $\chi'_u(G)$ be the smallest k such that Alice has a winning strategy.

Since Bob is allowed to miss his first turn, we have $\chi'_g(G) \le \chi'_u(G)$ for any graph G. Here we prove the following upper bound on $\chi'_u(G)$.

Theorem 4 *For every $\mu > 0$ there is $\varepsilon > 0$ such that any graph G with $\Delta(G) \ge (1/2 + \mu)v(G)$ satisfies*

$$\chi'_u(G) \le (2 - \varepsilon)\Delta(G). \tag{8}$$

The rest of Section 3 is dedicated to proving Theorem 4.

Let us first specify some notation we are going to use. We abbreviate an unordered pair $\{x, y\}$ as xy.

Suppose that we fix players' strategies and observe the game. Let us agree that we immediately stop the game if there is an uncolored edge incident to all possible colors. (Then no player will be able to color it and Bob automatically wins.) A *round* consists of a move of Bob (possibly skipped) followed by a move of Alice. Let A_r and B_r denote the sets of edges colored by Alice and Bob respectively after the initial r rounds. If the game ended earlier, before r full rounds were completed, let A_r and B_r denote the final edge-sets colored by Alice and Bob respectively.

Let $C_r = A_r \cup B_r$ consist of the colored edges after Round r. Let $c : E(G) \to [k]$ be the (possibly partial) coloring constructed at the end of the game. Let C'_r be the set of colored edges before Alice's move in Round r. For a vertex x of G, let $C_r(x)$ be the set of the colors of the C_r-edges incident to x:

$$C_r(x) = \{c(xy) : xy \in C_r\}.$$

The sets $A_r(x)$, $B_r(x)$, and $C'_r(x)$ are defined analogously.

Proof of Theorem 4. We will use various positive constants, whose dependences are as follows

$$\mu \gg c_1 \gg c_2 \gg c_3 \gg c_4 \gg c_5 \gg \varepsilon > 0,$$

where $a \gg b$ means that b is sufficiently small depending on a. It is enough to prove Theorem 4 for all sufficiently large n. Indeed, for any order-n graph G we have $\Delta(G) \le n$ and $\chi'_u(G) \le 2\Delta(G) - 1 \le (2 - 1/n)\Delta(G)$; thus the theorem becomes valid for every $n \le n_0$ if ε is reduced below $1/n_0$.

Let n be sufficiently large. Let the asymptotic notation, like $O(1)$, refer to the case that $n \to \infty$ while μ, c_1, etc, are fixed. Let G be an arbitrary graph of order n and maximum degree $d \ge (\frac{1}{2} + \mu)n$. Let $k = \lfloor (2 - \varepsilon)d \rfloor$.

Here is the strategy of Alice.

She makes two types of moves: *R-moves* (or *random moves*) and *S-moves* (or *set moves*). If Bob skipped his move, then Alice makes an R-move. An R-move consists of selecting an uncolored edge, uniformly at random from all uncolored edges of G. (The coloring rule, which is the same for both R-moves and S-moves, will be described shortly.) If Bob selected an edge xy in the previous move, then Alice throws a biased coin. With probability $1 - c_1$, she makes an R-move. With probability $c_1/2$, she makes an S-move at x, that is, picks a random uncolored edge at x. (If all edges at x have already been colored, then Alice makes an R-move instead.) With probability $c_1/2$, she makes an S-move at y, that is, picks a random uncolored edge at y (or makes an R-move if all edges at y have already been colored).

The rules for selecting Alice's edge uv are different for these two types of moves but the coloring rule is the same: the color $c(uv)$ is chosen uniformly at random from all admissible colors (that is, from the set $[k] \setminus (C'_r(u) \cup C'_r(v))$, where r is the number of the current round). There is always at least one available color for the edge uv, for otherwise we would have already stopped the game and declared Bob to be the winner. Let R_r and

S_r denote the sets of Alice's R-moves and S-moves respectively after r rounds. Thus, for every r,

$$A_r = R_r \cup S_r.$$

Note that if Bob has a winning strategy, then (since this is a complete information game) Bob has a *deterministic* winning strategy. Hence, in order to prove the theorem, it is enough to show that, for any fixed (deterministic) strategy of Bob, this random strategy of Alice has non-negative probability of winning.

So let us fix some strategy of Bob and let Alice play as above. Let

$$D = \{x \in V(G) : d(x) \geq (1 - 2\varepsilon)d\},$$

where $d(x)$ denotes the degree of x. Clearly, if we take an edge not entirely inside D, then at most $d + (1 - 2\varepsilon)d < k$ colors are forbidden, so this edge can always be colored. For $x \in D$ let

$$r(x) = \min\{r : |C_r(x)| \geq c_2 d\}. \tag{9}$$

Here is an informal description why Alice wins whp. We will show that whp two vertices of D will share at least εd common colors before the game ends. Indeed, if this occurs, then Alice wins because every edge gets colored: the number of forbidden colors is at most $(d-1)+(d-1)-\varepsilon d < k$. In fact, we show that this event occurs early in the game, after at most r rounds, where r can be set to be, for example, $4c_2 n^2$. Since the set C_r, containing at most $2r$ edges, is small, when Alice colors a random edge incident to a vertex $x \in D$ in some Round i, the color of this edge is spread on almost all the set $[k] \setminus C'_i(x)$. Hence, it is enough to show that whp each $|A_r(x)|$, $x \in D$ is fairly large. To this end observe that if $|B_r(x)|$ is small, then $|R_r(x)|$ is large because then any R-move had a chance at least $(d - |C_r(x)|)/\binom{n}{2}$ to pick an edge at x; otherwise $|S_r(x)|$ is large, being whp at least $(c_1/4) \times |B_r(x)|$. This is why we need an occasional S-move: to prevent Bob from claiming almost all edges at some vertex $x \in D$.

Let us present a rigorous proof. We define a family of 'bad' events and establish the following two properties. *Property I:* the expected value of the sum of the indicator functions of bad events is $o(1)$. *Property II:* if none of the bad events occurs then Alice necessarily wins. Then the theorem clearly follows.

All our bad events will be split into a few families. For each family we immediately analyze Property I, leaving the proof of Property II until the very end. For the notational convenience, we identify each event with its indicator function.

3.1 The First Family

Each event $B^1_{z,m,r}$ of this family is indexed by a triple (z,m,r), where $z\in D$ and $0\le r<m\le e(G)$. Informally speaking, if none of the events $B^1_{z,m,r}$ occurs, then the number of A-edges that hit a vertex $z\in D$ at any interval of the game is not much smaller than the expected value, provided that the final degree of z is not too big.

Here a formal definition of $B^1_{z,m,r}$. If the game ends before Round m, we set $B^1_{z,m,r}=0$. So suppose that the game continues for at least m rounds. We set $B^1_{z,m,r}=0$ if $C_m(z)\ge 4c_2d$ or if

$$|A_m(z)\setminus A_r(z)|\ge \frac{(m-r)(2-4c_1)}{n}-c_5n. \tag{10}$$

Otherwise, we set $B^1_{z,m,r}=1$.

Let us show that

$$\sum_{z\in D}\ \sum_{0\le r<m\le e(G)} E(B^1_{z,m,r})=o(1). \tag{11}$$

We fix z,m,r and estimate the probability of $B^1_{z,m,r}$. Consider a moment when Alice is about to make a move in Round i, with $r<i\le m$. Assume that z is currently incident to fewer than $4c_2d$ colored edges for otherwise we necessarily have $C_m(z)\ge 4c_2d$ and $B^1_{z,m,r}=0$. The probability of Alice's making an R-move is at least $1-c_1$, whether or not Bob skipped his previous move. We have at least $d(z)-4c_2d\ge(1-2\varepsilon-4c_2)d$ uncolored edges at z and at most $dn/2$ edges in total. Hence, the probability of increasing the number of A-edges at z is at least

$$(1-c_1)\,\frac{(1-2\varepsilon-4c_2)d}{dn/2}>\frac{2-4c_1}{n}=:p$$

at each round. Hence, if we assume that $C_i'(z)\le 4c_2d$ for each $i\in[r+1,m]$, then the left-hand side of (10) can be bounded from below by coupling with the $(m-r,p)$-Binomial variable. The Chernoff bound implies that the probability that (10) fails is exponentially small in n. Since the number of choices of the triple (z,m,r) is $O(n^5)$, the inequality (11) follows.

3.2 The Second Family

Here we define the event $B^2_{\{x,y\}}$, where $x,y\in D$, $x\ne y$. Using our convention, we will abbreviate it as B^2_{xy}. Roughly speaking, we observe the game for the initial r rounds for some r. Suppose that x and y have not acquired at least εd common colors yet.

Furthermore, suppose that one of them (say x) is incident to at least c_3d uncolored edges whose other endpoint does not see at least c_3d of the colors appearing at y. Then it is very unlikely that, in next c_4dn rounds, x gets almost none of the colors that were present at y at Round r.

Here is a formal definition of B^2_{xy}. If $B^1_{u,m,r} = 1$ for some $0 \le r < m \le e(G)$ and $u \in \{x, y\}$, then we immediately set $B^2_{xy} = 0$. So let us suppose otherwise. Let $r = \max(r(x), r(y))$, where $r(z)$ is the function defined by (9). If r is undefined (i.e. the game stops before each of x and y gets C_r-degree at least c_2d), then we set $B^2_{xy} = 0$. (It will be the case that some other bad events will be 'responsible' for this.) If

$$|C_r(x) \cap C_r(y)| \ge \varepsilon d, \tag{12}$$

then we set $B^2_{xy} = 0$, so let us suppose that (12) does not hold. We define

$$\begin{aligned} Z_{(x,y)} &= \{z : xz \in E(G) \setminus C_r,\ |C_r(y) \setminus C_r(z)| \ge c_3d\}, &(13)\\ Z_{(y,x)} &= \{z : yz \in E(G) \setminus C_r,\ |C_r(x) \setminus C_r(z)| \ge c_3d\}. &(14)\end{aligned}$$

If $\max(|Z_{(x,y)}|, |Z_{(y,x)}|) < c_3d$, then we set $B^2_{xy} = 0$. So suppose otherwise and let $\{u, v\} = \{x, y\}$ satisfy $|Z_{(u,v)}| \ge c_3d$. (If both assignments $u = x$ and $u = y$ work, we can agree that e.g. u is the smaller of x and y with respect to some fixed linear order on $V(G)$.)

Let us observe the game until Round m, where

$$m = r + \lfloor c_4dn \rfloor. \tag{15}$$

If the game ends before Round m, then we do the following: Set $B^2_{xy} = 1$ if the edge xy is uncolorable at the end (possibly one of a few uncolorable edges) and set $B^2_{xy} = 0$ otherwise (that is, if xy is colored or can be properly colored in the final position).

So, suppose that the game lasts at least until Round m. If

$$|C_m(x) \cap C_m(y)| \ge \varepsilon d, \tag{16}$$

we set $B^2_{xy} = 0$; otherwise, we set $B^2_{xy} = 1$. This finishes the description of the event B^2_{xy}.

Let us prove that

$$\sum_{\substack{x,y \in D\\ x \ne y}} E(B^2_{xy}) = o(1). \tag{17}$$

Let us fix $xy \in \binom{D}{2}$ and estimate the probability of B^2_{xy}. We analyze the game, starting from Round r and assuming that the previous development of the game does not rule out B^2_{xy} yet. In particular, we have defined u, v with $\{u, v\} = \{x, y\}$ and $|Z_{(u,v)}| \geq c_3 d$.

We will observe the game in Rounds $r + 1$ to m', where

$$m' = \min(m, r'), \tag{18}$$

where r' is the total number of the rounds until the game stops.

Since $m' - r \leq m - r \leq c_4 dn$, we have, for any $i \in [r + 1, m']$, that

$$|Z_{(u,v)} \setminus Z'_i| \leq 2 \times \frac{m - r}{c_3 d/2} \leq 4 \frac{c_4 dn}{c_3 d} \leq \frac{c_3 d}{4},$$

where we define $Z'_i = \{z \in Z_{(u,v)} : |C_r(v) \setminus C'_i(z)| \geq c_3 d/2\}$. Indeed, every vertex in $Z_{(u,v)} \setminus Z'_i$ must gain at least $c_3 d/2$ colored edges in Rounds $r + 1$ to i, resulting in the first inequality. The final inequality follows from $c_4 \ll c_3$. Thus

$$|Z'_i| \geq |Z_{(u,v)}| - \frac{c_3 d}{4} \geq \frac{3c_3 d}{4} \tag{19}$$

Let I consist of those $i \in [r+1, m']$ such that in Round i Alice colors an edge between u and Z'_i. Let E_0 be the event that B^2_{xy} occurs and $|I| \geq c_5 d$. Let us show that the probability of each of E_0 and $B^2_{xy} \setminus E_0$ (given the previous history up to Round r) is exponentially small in n—this will prove (17) because there are $O(n^2)$ choices of xy in total.

Let us analyze E_0 first. We will make use of the following coupling. Let $p_0 = (c_3/2 - \varepsilon)/2$. Let $X = (X_1, X_2, \dots)$ be an infinite 0/1-sequence where each entry is 1 with probability p_0, independently of the other entries. Initially we set $k = 1$.

Let us observe the rounds one by one as the game progresses. Let $i \geq r+1$ be number of the current round.

Suppose first that, in this Round i, Alice has selected and is about to color an edge uz with some $z \in Z'_i$. Let $W = [k] \setminus (C'_i(u) \cup C'_i(z))$ consist of all available colors for uz. Let $W' = W \cap C'_i(v)$ consist of those available colors that are also present at v. Alice increases the number of common colors at u and v with probability $p = |W'|/|W|$.

By the definition of Z'_i there are at least $c_3 d/2$ colors of $C_r(v) \subseteq C'_i(v)$ that are absent in $C'_i(z)$. Assume that at most εd of these colors are present in $C'_i(u)$ for otherwise (16) holds, $B^2_{xy} = 0$, and $E_0 = 0$. Hence, $|W'| \geq c_3 d/2 - \varepsilon d$ and $p = |W'|/|W| \geq (c_3 d/2 - \varepsilon d)/2d = p_0$.

Our coupling requires that the edge uz is colored with a color from W' whenever the k-th element X_k of X is 1. This can be achieved, for example, as follows. If $X_k = 1$, Alice picks a random color from W'. If $X_k = 0$, then Alice picks, with probability $(p - p_0)/(1 - p_0)$ a random color from W' and with probability $1 - (p - p_0)/(1 - p_0)$ a random color from $W \setminus W'$. This gives the uniform distribution on the set W of all available colors. Indeed, any two colors both from W' or from $W \setminus W'$ are equally likely to be picked while the probability of selecting a color from W' is exactly

$$p_0 + (1 - p_0)\,\frac{p - p_0}{1 - p_0} = p = \frac{|W'|}{|W|}.$$

Now, we increase k by 1 so that the new (unexposed) value of X_k is independent of the previous history. Continue the game.

If, in Round i, Alice does not color an edge uz with $z \in Z_i'$, then we do not do anything (and do not increase the counter k).

It follows that if E_0 occurs, then the first $\lfloor c_5 d \rfloor$ elements of X contain at most εd ones. By the Chernoff bound, this has exponentially small in n probability, giving the desired result. (Note that we do not have to take the union bound over all choices of I since we were feeding in the bits of X only when there was demand.)

Next, we analyze $E_0' = B_{xy}^2 \setminus E_0$, the event that B_{xy}^2 occurs and $|I| < c_5 d$. We split it further into two complementary sub-events $E_0' = E_1 \cup E_2$ depending respectively on whether or not Bob colors at least $c_3 d/4$ edges incident to u in Rounds $r+1$ to m', where m' is defined by (18).

In order to analyze E_1, consider the first $l = \lfloor c_3 d/4 \rfloor$ moves of Bob after Round r incident to u. Let us consider Alice's move in any such Round i, when Bob has just colored an edge at u. Of all G-edges between u and Z_i', at most $c_5 d$ edges are colored by Alice (we can assume this for otherwise $|I| \ge c_5 d$) and, trivially, at most $l \le c_3 d/4$ edges are colored by Bob. Hence, by (19), the probability that Alice picks an edge between u and Z_i' in Round i is at least

$$\frac{c_1}{2} \times \frac{3c_3 d/4 - c_5 d - c_3 d/4}{d} \ge \frac{c_1 c_3}{5}.$$

Similarly to above, we can couple this with an infinite 0/1-sequence $X = (X_1, X_2, \dots)$, whose each entry is 1 with probability $c_1 c_3/5$, where we read the next bit of X after those moves of Bob that touch u. It follows that if $E_1 = 1$, then there are less than $c_5 d$ ones among the first l members of X_1. The Chernoff bound shows that the probability of this (and thus of E_1) is exponentially small in n.

Let us analyze E_2.

Suppose first that $m' < m$ (and that $B^2_{xy} = 1$). Recall that m and m' are defined by (15) and (18). Then xy is uncolorable in the final position, so at least $(1-\varepsilon)d$ edges incident to u get colored. In particular, at least $|Z'_{m'}| - \varepsilon d$ edges between u and $Z'_{m'}$ are colored. Alice colors at most $c_5 d$ of these edges since E_0 does not occur, while Bob colors at most $c_3 d/4$ of these edges since E_1 does not occur. By (19), we conclude that $c_5 d + c_3 d/4 \geq 3c_3 d/4 - \varepsilon d$, a contradiction.

Hence, assume that $m' = m$, that is, the game lasts for at least m rounds. We observe $\lfloor c_4 nd \rfloor$ rounds after Round r and, in each Round i with $r < i \leq m$, the probability of Alice's hitting an edge between u and Z'_i is at least

$$(1-c_1) \times \frac{3c_3 d/4 - c_5 d - c_3 d/4}{nd/2}$$

because neither E_0 nor E_1 occurs. Again, the probability of fewer than $c_5 d$ successes (which is needed to avoid $|I| \geq c_5 d$) is exponentially small. This completely proves (17).

3.3 The Third Family

Its events B^3_a are indexed by a color $a \in [k]$. The event B^3_a occurs if and only if the graph R_m contains at least $37c_2 n$ edges of color a, where $m = \lfloor 4c_2 n^2 \rfloor$. (Recall that R_m consists of all R-moves of Alice made during the first m rounds.)

Initially, let us set $H = \emptyset$ and let X be an infinite 0/1-sequence where each entry is 1 with probability $3/n$ independently of the other entries. Let the game last for r' rounds.

We observe Alice's moves until Round m', where $m' = \min(m, r')$. Let us consider a moment when Alice has just selected an R-edge xy and is about to color it in some Round i. If there are less than $n/3$ available colors for the edge xy at the current moment, then we just add the pair xy to H and proceed with the game. Suppose that there are more than $n/3$ available colors. The probability of selecting the color a for $c(xy)$ is at most $3/n$. We read the next unexposed bit of X. Our coupling requires that if it is 0, then Alice does not select color a for $c(xy)$.

Consider the partial coloring right after Round m'. Let Y consist of vertices of G of $C_{m'}$-degree at least $n/3$. We have $|Y|(n/3) \leq 2m$; thus $|Y| \leq 24c_2 n$. Every edge xy of H has to intersect Y for otherwise the number of available colors at xy, even at Round m', is at least $(2-\varepsilon)d - n/3 - n/3 > n/3$, a contradiction. Since each color class is a matching, the number of color-a edges inside H is at most $|Y|$. It follows that if $B^3_a = 1$, then the first m entries of X contain at least $37c_2n - 24c_2n = 13c_2n$ ones. By the Chernoff bound this has probability exponentially small in n. Hence $\sum_{a=1}^{k} E(B_a^3) = o(1)$.

3.4 The Fourth Family

Let $0 \le m \le e(G)$.

The event B_m^4, occurs if and only if $|S_m| > c_1 m + c_3 n^2$. (Recall that S_m consists of Alice's S-moves made in the first m rounds.) Since the probability of increasing the current $|S_r|$ in any round is at most c_1, the Chernoff bound easily implies that $\sum_{m=0}^{e(G)} E(B_m^4) = o(1)$.

3.5 Putting All Together

Let us show that if none of the above bad events occurs, then Alice surely wins. Let us assume on the contrary that the game ends when an edge $xy \in E(G)$ cannot be properly colored. If there are a few choices for xy, pick one arbitrarily. (Recall that we stop the game as soon as an uncolorable edge appears.)

This means that $x, y \in D$ and each of x and y is incident to at least $(1-\varepsilon)d$ colored edges. Thus $r(x)$ and $r(y)$ are well-defined. Let $r = \max(r(x), r(y))$. Assume $r(x) \ge r(y)$. Thus $r = r(x)$ and $0 \le |C_r(x)| - c_2 d < 2$. Since xy cannot be properly colored at the end of the game, Inequality (12) is false, that is,

$$|C_r(x) \cap C_r(y)| < \varepsilon d. \tag{20}$$

Since $B_{x,r,0}^1 = 0$ and $|C_r(x)| \le c_2 d + 2 < 4c_2 d$, we have by (10) applied to the first r rounds that

$$c_2 d + 2 \ge |A_r(x)| \ge \frac{r(2-4c_1)}{n} - c_5 n.$$

It follows from $c_5, c_2 \ll c_1$ that, for example,

$$r \le \frac{(1+3c_1)c_2 n d}{2}. \tag{21}$$

Claim 1 $\max(|Z_{(x,y)}|, |Z_{(y,x)}|) \ge c_3 d$.

Proof of Claim. Suppose that the claim is not true. Let $l = |C_r(y)|$. Then, by the definition of $Z_{(x,y)}$, there are at least

$$d_G(x) - |C_r(x)| - |Z_{(x,y)}| \ge (1-2\varepsilon)d - (c_2 d + 2) - c_3 d \ge (1-2c_2)d,$$

vertices of G such that, after the r-th round, each sees at least $l - c_3 d$ colors from $C_r(y)$. Likewise, there are at least

$$d_G(y) - l - |Z_{(y,x)}| \geq (1-2\varepsilon)d - l - c_3 d \geq (1-2c_3)d - l$$

vertices of G, each seeing at least $|C_r(x)| - c_3 d \geq c_2 d - c_3 d$ colors from $C_r(x)$. This requires at least

$$\frac{1}{2}\Big((1-2c_2)d(l-c_3 d) + ((1-2c_3)d - l)(c_2 d - c_3 d)\Big) \geq \frac{dl}{2} + c_2 \frac{-3dl + d^2}{2} - c_2^2 n^2 \quad (22)$$

edges of C_r, each colored by a color in $C_r(x) \cup C_r(y)$. By (20) and since each color class contains at most $n/2$ edges, we double-count at most

$$|C_r(x) \cap C_r(y)| \times (n/2) \leq \varepsilon dn/2 \leq c_2^2 n^2 \quad (23)$$

edges. Moreover, by (21) we have $r \leq 4c_2 n^2$ and since no bad event B_a^3 occurs, A_r can contribute at most

$$|C_r(x) \cup C_r(y)| \times (37 c_2 n) \leq (c_2 d + 2 + l) \times (37 c_2 n)$$

to (22). Since B_r^4 does not occur, S_r contributes at most $c_1 r + c_3 n^2$ to (22). Finally, $|B_r| \leq r$. From (22) and (23), we obtain that

$$\frac{dl}{2} + c_2 \frac{-3dl + d^2}{2} - 2c_2^2 n^2 \leq (c_2 d + 2 + l) \times (37 c_2 n) + c_1 r + c_3 n^2 + r.$$

Using (21), we obtain (after routine simplifications) that

$$l\left(\frac{d}{2} - \frac{3c_2 d}{2} - 37 c_2 n\right) \leq -c_2 \frac{d^2}{2} + c_2 \frac{nd}{2} + 3c_1 c_2 n^2 \quad (24)$$

This is a contradiction to $l = |C_r(y)| \geq c_2 d$, $d \geq (\frac{1}{2} + \mu)n$, and $\mu \gg c_1 \gg c_2$. The claim is proved. ▮

So we can define $\{u, v\} = \{x, y\}$ as it is done after (14). Let $m = r + \lfloor c_4 dn \rfloor$. The game cannot end before Round m for then $B_{xy}^2 = 1$ (as xy is responsible for the end of the game). Again, since $B_{xy}^2 = 0$, we conclude that (16) holds. This means that, after Round m, x and y share at least εd colors, so there will always be a choice of color for xy. This contradicts our assumption and proves that Alice wins. This completes the proof of Theorem 4. ▮

Acknowledgments

The authors thank Małgorzata Bednarska for the very helpful discussions.

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