How to Play Dundee

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Abstract

We consider the following one-player game called *Dundee*. We are given a deck consisting of s_i cards of Value i, where $i = 1, \dots, v$, and an integer $m \le s_1 + \dots + s_v$. There are m rounds. In each round, the player names a number between 1 and v and draws a random card from the deck. The player loses if the named number coincides with the drawn value in at least one round.

The famous *Problem of Thirteen*, proposed by Montmort in 1708, asks for the probability of winning in the case when $v = 13$, $s_1 = \dots = s_{13} = 4$, $m = 13$, and the player names the sequence $1, \dots, 13$. This problem and its various generalizations were studied by numerous mathematicians, including J. and N. Bernoulli, De Moivre, Euler, Catalan, and others.

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However, it seems that nobody has considered which strategies of the player maximize the probability of winning. We study two variants of this problem. In the first variant, the player's bid in Round i may depend on the values of the random cards drawn in the previous rounds. We completely solve this version. In the second variant, the player has to specify the whole sequence of m bids in advance, before turning any cards. We are able to solve this problem when $s_1 = \cdots = s_v$ and m is arbitrary.

1 Introduction

1.1 Historical Remarks

The following *Game of Thirteen* (*jeu du treize*) was proposed by Montmort [25, Page 185] in 1708. Randomly shuffle the standard deck of 52 cards. For convenience, let us denote card values by numbers. Thus we have 13 different values $1, \ldots, 13$, each appearing 4 times. In Round i, where $i = 1, 2, \ldots, 13$, the player names Value i and deals a card from the remaining deck face up. If there is a *coincidence*, that is, the revealed card has the named value in at least one round, then the player loses. If there is no coincidence during the thirteen rounds, then the player wins. What is the probability of winning?

This problem had a great influence on the development of probability theory. We refer the reader to a nice survey by Takács [31], from where most of the authors' knowledge on the history of the problem comes.

A popular generalization, called the *Problem of Coincidences* (*jeu de rencontre*), is to consider decks with card values $1, \ldots, v$, each value repeated s times, and to study the number of coincidences. Various contributions to this problem were made by Montmort himself [25, 26], Johann Bernoulli (see [26, pp. 283–298]), Nikolaus Bernoulli (see [26, pp. 300–301 & 324]), De Moivre [24], Euler [10, 11], and others. Catalan [6] considered a further generalization where there are $m \leq v$ rounds and the player names the sequence $1, \ldots, m$. Greenwood [12], Kaplansky [16], Greville [13], and others made the first steps in the study of the version of the problem where the deck is not required to have the same number of cards of each value. Many introductory combinatorics or probability textbooks include a treatment of some version of the problem. Scientific articles on the topic (mostly of expository nature) still keep appearing, the more recent ones including Penrice [27], Cameron and Cohen [5], Boston et al. [3], Clarke and Sved [7], Doyle, Grinstead, and Snell [9], Knudsen and

Skau [18], Michel [22], Linnell [19], Sanchis [28], Kessler and Schiff [17], Avenhaus [2], Manstavičius [21], Diaconis, Fulman, and Guralnik [8]. (The annotated on-line bibliography [30] maintained by Torsten Sillke was very helpful in compiling this list.)

However, it seems (as far as we could see) that nobody has systematically studied the version where the player has the freedom to choose the value to be named in each round and aims at maximizing the probability of winning. Here we try to fill this gap. Let us formalize the problem first.

1.2 Some Definitions

For integers $n \geq m \geq 1$, let us denote $[m,n] = \{m, m+1, \ldots, n-1, n\}$ and $[n] = [1,n] = \{1, \ldots, n\}$. Let the cards in the deck assume possible values $1, \ldots, v$ and, for $i \in [v]$, let s_i be the number of cards of Value i. We call such a collection of cards the $(s_1, \ldots, s_v)$-*deck* and we call the sequence $\mathbf{s} = (s_1, \ldots, s_v)$ the *composition vector* or simply the *composition* of the deck. Let $\Sigma(\mathbf{s}) = s_1 + \cdots + s_v$ be the total number of cards. For example, the standard 52-card deck can be described as the $(4, \ldots, 4)$-deck where 4 is repeated 13 times. We do not require that $s_1 = \cdots = s_v$ in general. Let an integer $m \leq \Sigma(\mathbf{s})$ be given.

In the *m-round* $\mathbf{s}$*-game*, the $\mathbf{s}$-deck is randomly shuffled, there are m rounds, and in each round the player names a card value (which we call a *bid*) and then deals one card from the remaining deck face up. The player loses if there is at least one coincidence in Rounds 1 to m. We assume that the player knows the integer m and the composition of the deck (that is, the sequence $(s_1, \ldots, s_v)$) in advance.

Of course, the outcome of the game depends not only on the player's strategy but also on the (random) order of the cards in the deck. Here we assume that the shuffling is *uniform*, that is, all card orderings are equally likely. We look for strategies that maximize the probability that the player wins.

Our initial interest in this problem came from the book by Harbin [14, Page 136], where he described the special case of the above game, namely, when $\mathbf{s} = (4, \ldots, 4)$ gives the standard 52-card deck and $m = 52$. Harbin calls this game *Dundee*, a name that we will use for the general case as well.

There are two versions of the problem depending on whether or not the player's bid in Round i may depend on the random values that appeared in the previous rounds. If this is allowed, then we call such strategies *adaptive*; otherwise we call them *advance*. Let us discuss these two cases separately.

1.3 Adaptive Strategies

Here the player remembers all the cards that have been dealt so far and thus knows all the remaining cards (but, of course, not their order). Then there is an intuitively obvious choice for his next bid: *name a value that appears the least number of times in the remaining deck.* We call a strategy that adheres to this rule at every round *greedy.* It is clear that, once the first k cards are exposed, the order of the remaining $\Sigma(\mathbf{s}) - k$ cards is still uniform. So, if there is the same number of the remaining cards of Values i and j, then guessing either of these two values leads, up to a symmetry, to the same game tree (with the same branching probabilities). In particular, any two greedy strategies have the same chances of winning in the m-round game. So, by a slight abuse of language, we call any such strategy *the greedy strategy.*

Clearly, the greedy strategy has the largest chances of surviving the next step, but this does not necessarily give the highest probability of winning in the whole game. For example, there might be another strategy performing worse in the first step, but resulting in better positions on the condition that the player has survived the first step. The latter situation is not an abstract speculation; in fact, it almost takes place in Dundee. For example, it is easy to show that if $v = 2$ and $m = s_1 + s_2$, then any strategy not missing a sure win is optimal (and so is as good as the greedy strategy). In fact, the case $v = 2$ is somewhat pathological: the probability of the player's winning in the general m-round case depends only on how often each value is called but not on the order in which this is made.

Proposition 1 *Let $s_1 \geq s_2 \geq 0$ and $1 \leq m \leq s_1 + s_2$. Let the player name Value 1 (resp. 2) b_1 (resp. b_2) times during $b_1 + b_2 = m$ rounds.*

Then the probability of winning is $\binom{s_1+s_2-b_1-b_2}{s_1-b_2}\binom{s_1+s_2}{s_1}^{-1}$. (Note that this is non-zero if and only if $b_1 \leq s_2$ and $b_2 \leq s_1$.)

In particular, if $m \leq s_1 - s_2$, then the (unique) optimal strategy is to name Value 2 all the time. Otherwise, the optimal strategies are exactly those for which the numbers $s_1 - b_2$ and $s_2 - b_1$ differ by at most 1.

However, the following result states that the greedy strategy strictly beats any other strategy when there are at least three different card values. In particular, the set of optimal bids in each round does not depend on the number of the remaining rounds.

Theorem 2 *Let $v \geq 3$ and $\mathbf{s} = (s_1, \ldots, s_v)$ be an arbitrary vector whose entries are non-negative integers. Let $m \leq \Sigma(\mathbf{s})$. Then the greedy strategy is the unique optimal strategy for the m-round $\mathbf{s}$-game.*

The proofs of these results and some further observations about the greedy strategy can be found in Section 2.

Unfortunately, it seems that there is no general closed formula for $g_m(\mathbf{s})$, the probability that the greedy strategy wins the m-round $\mathbf{s}$-game. However, there is an obvious recurrence relation for computing $g_m(\mathbf{s})$, namely Identity (3) here, that can be used to determine $g_m(\mathbf{s})$ for some small $\mathbf{s}$. The computer code written by the authors (available from [20]) shows that

$$g_{52}(\underbrace{4,\ldots,4}_{13\ \text{times}}) = \frac{4705858489851502066775082587 2}{174165229296062536531664039375} = 0.27019... \tag{1}$$

As we see from (1) the probability of winning in Dundee for the standard deck is not too small, more than 27%. However, Harbin [14, Page 136] writes: "*I have tried to do this and have not yet managed to deal right through the pack; it is quite amazing how impossible it is.*" It is conceivable that Harbin used some strategy similar to greedy but the discrepancy to (1) comes from not keeping track of the dealt cards.

Finally, the problem of finding the strategies that *minimize* the probability of winning turns out to be easy and the answer is provided by the following result. Let us call a situation in the game, when the player is about to name a bid, *decided* if $m' > s'_1 + \cdots + s'_v - \max(s'_1, \ldots, s'_v)$, where s'_i is the number of the remaining cards of Value i and m' is the number of the remaining rounds. Otherwise, the situation is *undecided*.

Theorem 3 *Let $v \geq 2$, $s_1 \geq \cdots \geq s_v \geq 1$, and $\mathbf{s} = (s_1, \ldots, s_v)$. Let $c = \Sigma(\mathbf{s})$ and let $m \leq c$.*

The minimum probability of winning is 0 if and only if the initial position is decided (that is, if $m > c - s_1$). Moreover, the strategies that surely lose are precisely those strategies for which a position that is undecided can never appear.

If $m \leq c - s_1$, then the smallest probability of winning is $\prod_{i=0}^{m-1} \frac{c-s_1-i}{c-i}$ and all strategies achieving it are anti-greedy *(always, name a most frequent remaining card or, equivalently, a card that occurs s_1 times in the remaining deck).*

1.4 Advance Strategies

Here it is required that the player's bid does not depend on the values of the previously turned cards. Clearly, the player can just name his whole sequence in advance and then start dealing cards. So we call such strategies *advance*. The strategy of the Game of Thirteen is an example of an advance strategy.

Since the order in which the values are named does not matter, we encode any advance strategy by the *bid vector* $\mathbf{b} = (b_1, \ldots, b_v)$, where b_i is the number of times that Value i is named. The entries of $\mathbf{b}$ are non-negative integers satisfying $\Sigma(\mathbf{b}) = m$. Let $\mathbf{Pr}(\mathbf{b}, \mathbf{s})$ be the probability that the advance bid $\mathbf{b}$ wins the m-round $\mathbf{s}$-game.

Problem 4 (Advance Bid Problem) *Given a composition vector* $\mathbf{s} = (s_1, \ldots, s_v)$ *and an integer* $m \leq \Sigma(\mathbf{s})$*, find all vectors* $\mathbf{b} = (b_1, \ldots, b_v)$ *that maximize* $\mathbf{Pr}(\mathbf{b}, \mathbf{s})$ *among all vectors with non-negative integer entries summing up to* m*.*

Let $c = \Sigma(\mathbf{s})$ be the total number of cards. Given a vector $\mathbf{b}$ with $\Sigma(\mathbf{b}) = m \leq c$, it is sometimes convenient to add to $\mathbf{b}$ extra $c - m$ bids of Value 0 that never causes a coincidence and to play the game for all c rounds. Then $c!\,\mathbf{Pr}(\mathbf{b}, \mathbf{s})$ is exactly the permanent of the $c \times c$-matrix $M(\mathbf{b}, \mathbf{s})$ whose entries are 0 and 1 depending of whether the bid corresponding to the row and the card value corresponding to the column are the same or not. Thus Problem 4 is somewhat reminiscent of the famous Minc Conjecture [23] proved by Brègman [4] (see also Schrijver [29] for a short proof) that asks for the maximum of the permanent of a 0/1 square matrix with given row-sums. In our problem, if we have $s_1 = \cdots = s_v = s$, then m row sums in $M(\mathbf{b}, \mathbf{s})$ are equal to $c - s$ and $c - m$ row sums are c. But, of course, we maximize the permanent over 0/1-matrices of a special type only and these two problems seem to be different in flavor.

The case of Problem 4 when the set $I = \{i : s_i = 0\}$ is non-empty is trivial: the optimal bids are precisely those bids $(b_1, \ldots, b_v)$ with $b_i = 0$ whenever $i \notin I$. Also, if $v = 2$, then Proposition 1 happens to answer Problem 4 as well (because the probability of winning in the cases covered by Proposition 1 depends only on how many times each value is named).

The *regular* deck (that is, the case when $s_1 = \cdots = s_v = s$) seems to be the most interesting and natural case. Intuition tells us that any optimal m-round bid should be *almost regular*, that is, it should name each value nearly the same number of times, $\lfloor m/v \rfloor$ or $\lceil m/v \rceil$. (Clearly, such a vector is unique up to a permutation of card values.) We prove that this is indeed true except the deck $(1, 1, 1)$ is somewhat exceptional: there are other bids that perform as well as the regular bid.

Theorem 5 *Let* $v \geq 3$*,* $\mathbf{s} = (s, \ldots, s)$ *be a regular* v*-vector, and* $m \leq sv$*. If* $\mathbf{s} = (1, 1, 1)$ *and* $m = 3$*, then there are* 7 *optimal advance bids for the*

$\mathbf{s}$*-deck:* $(1,1,1)$ *and the permutations of* $(2,1,0)$*. Otherwise, the optimal advance bids are precisely almost regular* v*-vectors with sum* m*.*

Thus, the bid vector $(1,\ldots,1)$ which corresponds to the player's sequence $1,2,\ldots,13$ in Montmort's Game of Thirteen does maximize the probability of winning (as well as Catalan's bid $1,\ldots,m$).

Unfortunately, a complete solution to Problem 4 for an arbitrary deck $\mathbf{s}$ has evaded us although some further results are presented in Section 3. We have written a computer program for determining $\mathbf{Pr}(\mathbf{b},\mathbf{s})$, see [20]. Table 2 of Section 3 lists all optimal advance bids for some small decks. One can spot some patterns and our proof techniques may be applicable to some other cases than those covered by Theorem 5. However, this problem in full generality remains open. In fact, we do not know if there is an algorithm that on input $\mathbf{s}=(s_1,\ldots,s_v)$ produces all optimal advance bids (or even just one) for the $\mathbf{s}$-deck with running time polynomial in $v\max(\log s_1,\ldots,\log s_v)$ (or even in $c=\Sigma(\mathbf{s})$). For general $c\times c$-matrices, Valiant [32] showed that the problem of computing the permanent is $\#\mathcal{P}$-complete (thus there is no polynomial time algorithm for the corresponding decision problem unless $\mathcal{P}=\mathcal{NP}$) while Jerrum, Sinclair, and Vigoda [15] presented an algorithm that outputs an arbitrarily close approximation in time that depends polynomially on c and the desired error.

The standard 52-card deck is covered by Theorem 5. Our code shows that the (unique) optimal advance bid for the 52-round game of naming each value 4 times wins with probability

$$\frac{4610507544750288132457667562311567997623087869}{284025438982318025793544200005777916187500000000}=0.01623..., \quad (2)$$

that is, the player wins in approximately 1 in 61.6 games. So the name *Frustration Solitaire* coined by Doyle, Grinstead and Snell [9] is not surprising. Doyle et al [9] obtained the same numerical answer as in (2). This is reassuring since they used a different method (the Principle of Inclusion-Exclusion) to derive (2).

Finally, the solution to the problem of *minimizing* the chances of the player's winning easily follows from Hall's Marriage Theorem and our Theorem 3.

Corollary 6 *Let* $v\geq 2$*,* $s_1\geq\cdots\geq s_v\geq 1$*, and* $\mathbf{s}=(s_1,\ldots,s_v)$*. Let* $c=\Sigma(\mathbf{s})$ *and let* $m\leq c$*. We minimize* $\mathbf{Pr}(\mathbf{b},\mathbf{s})$ *over all bid* v*-vectors* $\mathbf{b}$ *with* $\Sigma(\mathbf{b})=m$*.*

The minimum is 0 *if and only if* $m>c-s_1$*. It is achieved by* $\mathbf{b}$ *if and only if there is some* $i\in[v]$ *with* $b_i>c-s_i$*.*

If $m \le c - s_1$, then the minimum is $\prod_{i=0}^{m-1} \frac{c-s_1-i}{c-i}$. It is achieved by $\mathbf{b}$ *if and only if there is an index $j \in [v]$ such that $s_j = s_1$ and $b_j = m$ (while $b_i = 0$ for all $i \in [v] \setminus \{j\}$).* ∎

2 The Greedy Strategy

2.1 The Case $v = 2$

Recall that *the greedy strategy* always chooses a value that is least frequent among the remaining cards. (In particular, it does not miss a sure win if all cards of some value have been already dealt out.) Let us prove Proposition 1 for a warm-up.

Proof of Proposition 1. First, let us prove that if $m = s_1 + s_2$ then any strategy succeeds with probability at most $\binom{s_1+s_2}{s_1}^{-1}$. We use induction on $s_1 + s_2$. This upper bound is trivially true if $\min(s_1, s_2) = 0$ so suppose otherwise. Let the player name, for example, Value 1 in the first round. Then he survives the first step with probability $\frac{s_2}{s_1+s_2}$; in this case the remaining cards form a uniformly shuffled $(s_1, s_2 - 1)$-deck. The induction assumption implies that the total probability of winning is at most $\frac{s_2}{s_1+s_2}\binom{s_1+s_2-1}{s_1}^{-1} = \binom{s_1+s_2}{s_1}^{-1}$, finishing the inductive step.

Also, any strategy that does not miss a sure win achieves this bound since then all inequalities in the above proof become equalities. On the other hand, if for some strategy there is a feasible situation where it goofs the case $\min(s_1, s_2) = 0$, then we can strictly improve the strategy by changing its behavior in this situation into a sure win (and using the old strategy in all other cases). So such a strategy cannot be optimal. This completely proves the case $m = s_1 + s_2$ of Proposition 1.

Finally, assume that $m = b_1 + b_2 < s_1 + s_2$ with $b_1 \le s_2$ and $b_2 \le s_1$. Let the player name Values 1 and 2 respectively b_1 and b_2 times during the first m rounds. Let P be the probability that this strategy wins the m-round game. If we condition on this, then the remaining deck has composition $(s_1 - b_2, s_2 - b_1)$. If the player is to continue playing (for example, greedily), then our previous argument for $m = s_1 + s_2$ implies that the probability of no coincidence at all is $P \times \binom{(s_1-b_2)+(s_2-b_1)}{s_1-b_2}^{-1}$. By the same token, this probability equals also $\binom{s_1+s_2}{s_1}^{-1}$. Indeed, for $i = 1, 2$, the condition $b_{3-i} \le s_i$ guarantees that if all cards of Value i have been dealt out, then the strategy has already exhausted all bids of Value $3 - i$ (and we have, in fact, $b_{3-i} = s_i$), and so this strategy does not miss a sure win. Hence, $P = \binom{s_1+s_2-b_1-b_2}{s_1-b_2}\binom{s_1+s_2}{s_1}^{-1}$, as required.

Finally, all remaining claims of Proposition 1 follow from the symmetry and unimodality of the sequence $\binom{s_1+s_2-m}{i}$, when i ranges from 0 to $s_1 + s_2 - m$. ∎

2.2 The Greedy Strategy is Optimal

Here, we show that the greedy strategy is the unique optimal strategy if $v \geq 3$. The main difficulty is to find suitable statements amenable to induction. Once these are found, the proof, although somewhat lengthy, essentially takes care of itself.

We need to introduce some notation and prove a few auxiliary results first. If a sequence has v entries, we call it a *v-sequence*. Let $\mathcal{V}_{v,c}$ consist of all non-increasing v-sequences of non-negative integers with sum c. From here until the proof of Theorem 2 (inclusive), we will always assume that the entries of composition vectors are ordered non-increasingly. The *i-th partial sum* of $\mathbf{s}$ is

$$\Sigma_i(\mathbf{s}) = s_1 + \cdots + s_i.$$

We will need the following operation: if $s_i \geq 1$, then $\mathbf{s}^i$ is the vector obtained from $\mathbf{s}$ by decreasing the i-th entry by 1 and reordering the new vector in the non-increasing manner (which is needed when $i < v$ and $s_i = s_{i+1}$).

Let $\mathbf{s} \in \mathcal{V}_{v,c}$ and $0 \leq m \leq c$. The function $g_m(\mathbf{s})$, which is the probability that the greedy strategy wins on the m-round $\mathbf{s}$-game, satisfies the following relations. If the last entry s_v is zero or if $m = 0$, then $g_m(\mathbf{s}) = 1$. Otherwise,

$$g_m(\mathbf{s}) = \sum_{i=1}^{v-1} \frac{s_i}{c}\, g_{m-1}(\mathbf{s}^i). \tag{3}$$

Indeed, the greedy strategy names s_v in the first round while s_i/c is the probability that the first random card has Value i in which case the remaining $c - 1$ cards form the uniformly shuffled $\mathbf{s}^i$-deck.

Let $\mathbf{q}, \mathbf{s} \in \mathcal{V}_{v,c}$. We say that $\mathbf{s}$ *majorizes* $\mathbf{q}$ (and write this as $\mathbf{s} \succeq \mathbf{q}$) if $\Sigma_i(\mathbf{s}) \geq \Sigma_i(\mathbf{q})$ for every $i \in [v-1]$. (Recall that by the definition of $\mathcal{V}_{v,c}$, $\Sigma_v(\mathbf{s}) = \Sigma_v(\mathbf{q}) = c$.)

For $\mathbf{s} \in \mathcal{V}_{v,c}$ and $\mathbf{q} \in \mathcal{V}_{v,d}$, let $P(\mathbf{s}, \mathbf{q})$ be the product over all $i \in [v]$ for which $s_i > q_i$ of $s_i(s_i - 1) \ldots (q_i + 2)(q_i + 1)$. We agree that if $q_i \geq s_i$ for each $i \in [v]$, then $P(\mathbf{s}, \mathbf{q}) = 1$. Note that $P(\mathbf{s}, \mathbf{q})$ is in general different from $P(\mathbf{q}, \mathbf{s})$ and that $P(\mathbf{s}, \mathbf{q})$ is always strictly positive.

Lemma 7 *If* $\mathbf{q}, \mathbf{s} \in \mathcal{V}_{v,c}$ *and* $\mathbf{q} \preceq \mathbf{s}$, *then*

$$P(\mathbf{q}, \mathbf{s}) \leq P(\mathbf{s}, \mathbf{q}). \tag{4}$$

Moreover, if $\mathbf{q} \neq \mathbf{s}$, *then the inequality is strict.*

Proof. We use induction on $c + v$. The base cases are $c \in \{0, 1\}$ and v arbitrary or $v = 1$ and c is arbitrary. In either case the equality $\Sigma_v(\mathbf{q}) = \Sigma_v(\mathbf{s}) = c$ implies that $\mathbf{s} = \mathbf{q}$ so there is nothing to do. So suppose that $\min(v, c) > 1$ and the validity of the lemma has been verified for all pairs (v, c) with a smaller sum.

Case 1 There is an index $i \in [v - 1]$ such that $\Sigma_i(\mathbf{q}) = \Sigma_i(\mathbf{s})$.

Fix any such index i. Let $\mathbf{q}' = (q_1, \ldots, q_i)$, $\mathbf{q}'' = (q_{i+1}, \ldots, q_v)$, $\mathbf{s}' = (s_1, \ldots, s_i)$, and $\mathbf{s}'' = (s_{i+1}, \ldots, s_v)$. Our assumptions imply that the sequences $\mathbf{s}'$ and $\mathbf{q}'$ (resp. $\mathbf{s}''$ and $\mathbf{q}''$) have the same sum c' (resp. c'') and length i (resp. $v - i$). By the assumption of Case 1, we have that for any $j \in [v - i - 1]$,

$$\Sigma_j(\mathbf{q}'') = \Sigma_{i+j}(\mathbf{q}) - \Sigma_i(\mathbf{q}) \leq \Sigma_{i+j}(\mathbf{s}) - \Sigma_i(\mathbf{s}) = \Sigma_j(\mathbf{s}''),$$

so $\mathbf{q}'' \preceq \mathbf{s}''$. Also, $\mathbf{q}' \preceq \mathbf{s}'$. Since by concatenating $\mathbf{q}'$ and $\mathbf{q}''$ (resp. $\mathbf{s}'$ and $\mathbf{s}''$) we obtain the non-increasing sequence $\mathbf{q}$ (resp. $\mathbf{s}$), we have

$$P(\mathbf{s}, \mathbf{q}) = P(\mathbf{s}', \mathbf{q}')P(\mathbf{s}'', \mathbf{q}''), \tag{5}$$

$$P(\mathbf{q}, \mathbf{s}) = P(\mathbf{q}', \mathbf{s}')P(\mathbf{q}'', \mathbf{s}''). \tag{6}$$

Since the length of each $\mathbf{q}'$ and $\mathbf{q}''$ is strictly smaller than v (while the sums c', c'' are at most c) the induction hypothesis applies to the pairs $(\mathbf{s}', \mathbf{q}')$ and $(\mathbf{s}'', \mathbf{q}'')$ and gives the required by (5) and (6). Moreover, if $\mathbf{q} \neq \mathbf{s}$, then $\mathbf{q}' \neq \mathbf{s}'$ or $\mathbf{q}'' \neq \mathbf{s}''$, and (4) is strict by the induction assumption.

Case 2 Not Case 1.

In particular, we have $s_1 \geq q_1 + 1 \geq 1$ and $q_v \geq s_v + 1 \geq 1$. Recall that $\mathbf{s}^i$ is the sequence obtained from $\mathbf{s}$ by decreasing the i-th entry by one and reordering the terms. The sequences $\mathbf{q}^v$ and $\mathbf{s}^1$ of non-negative integers have the same length v and sum $c - 1$. Also, $\mathbf{s}^1 \succeq \mathbf{q}^v$ because we are not in Case 1 (and thus $\Sigma_i(\mathbf{s}) \geq \Sigma_i(q) + 1$ for every $1 \leq i \leq v - 1$). Using the induction assumption and the inequalities $q_v > s_v$ and $s_1 > q_1$, we obtain

$$\frac{P(\mathbf{q}, \mathbf{s})}{q_v} = P(\mathbf{q}^v, \mathbf{s}^1) \leq P(\mathbf{s}^1, \mathbf{q}^v) = \frac{P(\mathbf{s}, \mathbf{q})}{s_1}.$$

Now, the required (strict) bound follows from $s_1 > q_1 \geq q_v$. ∎

Lemma 8 *For any sequences* $\mathbf{q}, \mathbf{s} \in \mathcal{V}_{v,c}$ *and any* $i \in [v]$ *such that* $s_i \geq 1$*, we have*

$$P(\mathbf{q}, \mathbf{s}^i) P(\mathbf{s}, \mathbf{q}) = s_i P(\mathbf{q}, \mathbf{s}) P(\mathbf{s}^i, \mathbf{q}). \tag{7}$$

Proof. Let j be the maximum index such that $s_j = s_i$ (possibly $j = i$). Since $\mathbf{s}^i = \mathbf{s}^j$, it is enough to prove the lemma for $\mathbf{s}^j$. Note that we do not have to reorder terms when we compute $\mathbf{s}^j$. If $q_j \geq s_j$, then

$$\begin{aligned} P(\mathbf{s}, \mathbf{q}) &= P(\mathbf{s}^j, \mathbf{q}) \\ P(\mathbf{q}, \mathbf{s}^j) &= s_j P(\mathbf{q}, \mathbf{s}). \end{aligned}$$

Otherwise (if $q_j < s_j$) we have

$$\begin{aligned} P(\mathbf{s}, \mathbf{q}) &= s_j \, P(\mathbf{s}^j, \mathbf{q}) \\ P(\mathbf{q}, \mathbf{s}^j) &= P(\mathbf{q}, \mathbf{s}). \end{aligned}$$

By multiplying these identities, we obtain the required equality in either case. ∎

Lemma 9 *For any sequences* $\mathbf{s}, \mathbf{q} \in \mathcal{V}_{v,c}$ *with* $\mathbf{q} \preceq \mathbf{s}$ *and any* $m \leq c$*, we have*

$$P(\mathbf{q}, \mathbf{s}) g_m(\mathbf{s}) \leq P(\mathbf{s}, \mathbf{q}) g_m(\mathbf{q}). \tag{8}$$

Moreover, if additionally $v \geq 3$ *and* $\mathbf{q} \neq \mathbf{s}$*, then the inequality in (8) is strict.*

Proof. We use induction on $c + v$. If $c \in \{0, 1\}$ or if $v = 1$, then $\mathbf{s} = \mathbf{q}$ and there is nothing to do. If $m = 0$, then we are done by Lemma 7. So suppose that $\min(c, v) > 1$ and $m \geq 1$.

Let $I = \{i \in [v-1] : s_i \geq 1\}$. Note that $v \notin I$ by the definition. The assumption $\mathbf{q} \preceq \mathbf{s}$ implies that $q_i \geq 1$ for every $i \in I$. Thus $\mathbf{q}^i$ and $\mathbf{s}^i$ are well-defined when $i \in I$. By a version of (3) that also works in the case $s_v = 0$, we have

$$\begin{aligned} P(\mathbf{q}, \mathbf{s}) g_m(\mathbf{s}) &= \frac{P(\mathbf{q}, \mathbf{s})}{c} \sum_{i \in I} s_i g_{m-1}(\mathbf{s}^i) \\ P(\mathbf{s}, \mathbf{q}) g_m(\mathbf{q}) &\geq \frac{P(\mathbf{s}, \mathbf{q})}{c} \sum_{i \in I} q_i g_{m-1}(\mathbf{q}^i). \end{aligned}$$

The inequality (8) will follow if we show that for every $i \in I$ we have

$$P(\mathbf{q}, \mathbf{s}) s_i g_{m-1}(\mathbf{s}^i) \leq P(\mathbf{s}, \mathbf{q}) q_i g_{m-1}(\mathbf{q}^i). \tag{9}$$

Claim 1 $\mathbf{q}^i \preceq \mathbf{s}^i$ for every $i \in I$.

Proof of Claim. Take any $i \in T$. Let h (resp. j) be the maximum index such that $q_h = q_i$ (resp. $s_j = s_i$). Then $\Sigma_f(\mathbf{q}) - \Sigma_f(\mathbf{q}^i)$ is 0 if $f \in [h-1]$ and is 1 if $h \le f \le v$. The analogous claim holds for $\mathbf{s}$.

Suppose that this index i violates Claim 1. This is possible only if $h > j$ and there is an $f \in [j, h-1]$ such that

$$\Sigma_f(\mathbf{q}) = \Sigma_f(\mathbf{s}). \tag{10}$$

If $f > j$, then $\Sigma_{f-1}(\mathbf{q}) \le \Sigma_{f-1}(\mathbf{s})$ and (10) imply that $s_f \le q_f = q_i$. Since $s_{f+1} \le s_f \le q_i = q_{f+1}$, in order to prevent the contradiction $\Sigma_f(\mathbf{q}) + q_{f+1} > \Sigma_f(\mathbf{s}) + s_{f+1}$, we have to assume that $s_f = q_f$. Thus, we can decrease f by one without violating (10). By iterating this argument, we can assume that $f = j$.

By (10) and $\Sigma_{j+1}(\mathbf{s}) \ge \Sigma_{j+1}(\mathbf{q})$, we have $s_{j+1} \ge q_{j+1}$. By the definition of j and h and the inequality $h > j$, we have $s_j > s_{j+1} \ge q_{j+1} = q_j$. We conclude, again by (10), that

$$\Sigma_{j-1}(\mathbf{q}) = \Sigma_j(\mathbf{q}) - q_j > \Sigma_j(\mathbf{q}) - s_j = \Sigma_{j-1}(\mathbf{s}),$$

a contradiction which proves the claim. ▌

Let $i \in I$ be arbitrary. By Claim 1, we can apply induction to $(\mathbf{s}^i, \mathbf{q}^i)$ and $m-1$, obtaining

$$P(\mathbf{s}^i, \mathbf{q}^i) g_{m-1}(\mathbf{q}^i) \ge P(\mathbf{q}^i, \mathbf{s}^i) g_{m-1}(\mathbf{s}^i). \tag{11}$$

Lemma 8 (applied twice) gives (7) and the identity $q_i P(\mathbf{q}^i, \mathbf{s}^i) P(\mathbf{s}^i, \mathbf{q}) = P(\mathbf{s}^i, \mathbf{q}^i) P(\mathbf{q}, \mathbf{s}^i)$. By multiplying these two identities, we obtain

$$\frac{P(\mathbf{s}, \mathbf{q}) q_i}{P(\mathbf{s}^i, \mathbf{q}^i)} = \frac{P(\mathbf{q}, \mathbf{s}) s_i}{P(\mathbf{q}^i, \mathbf{s}^i)}. \tag{12}$$

By multiplying (11) and (12) we obtain the required inequality (9). This proves (8).

Finally, let us assume that $v \ge 3$ and $\mathbf{q} \ne \mathbf{s}$. Suppose that $m > 0$, for otherwise (8) is strict by Lemma 7 and we are done. In order to show that (8) is strict it is enough to show that (11) is strict for at least one $i \in I$. By induction, it is enough to find an $i \in I$ such that $\mathbf{q}^i \ne \mathbf{s}^i$.

If there is an $i \in I$ such that $\Sigma_i(\mathbf{s}) \ge \Sigma_i(\mathbf{q}) + 2$, then $\mathbf{s}^1 \ne \mathbf{q}^1$ and we are done.

So, suppose that $\Sigma_i(\mathbf{s}) \le \Sigma_i(\mathbf{q}) + 1$ for every $i \in I$. We cannot have $\Sigma_i(\mathbf{s}) = \Sigma_i(\mathbf{q})$ for all $i \in I$ for otherwise $s_i = q_i$ for every $i \in I$, but then

$\Sigma(\mathbf{s}) = \Sigma(\mathbf{q})$ implies that $\mathbf{s} = \mathbf{q}$, contradicting our assumption. Let $j \in I$ be the smallest index such that $s_j \neq q_j$. It follows that $s_j = q_j + 1$. If $\mathbf{s}^j \neq \mathbf{q}^j$, then we are done, so suppose otherwise. We have $\Sigma_j(\mathbf{s}) = \Sigma_j(\mathbf{q}) + 1$ and $\Sigma_j(\mathbf{s}^j) = \Sigma_j(\mathbf{q}^j)$. It follows that $q_{j+1} = q_j = s_j - 1$ and $s_{j+1} < s_j$. If $j + 1 \in I$ then we are done by applying the induction assumption to $\mathbf{s}^{j+1} \neq \mathbf{q}^{j+1}$.

So, suppose that $j + 1 \notin I$. There are two possible reasons for this. Suppose first that $j < v$ and $s_{j+1} = 0$. We cannot have $q_{j+1} = 0$ for otherwise $q_{j+2} = \cdots = q_v = 0$ and $\Sigma(\mathbf{q}) = \Sigma(\mathbf{s}) - 1$. Also, $q_{j+1} < 2$ for otherwise $\Sigma_{j+1}(\mathbf{s}) < \Sigma_{j+1}(\mathbf{q})$. Thus $q_{j+1} = 1$, which in turn implies that $q_j = 1$ and $s_j = 2$. But now, in view of $v \geq 3$, we have $\mathbf{q}^1 \neq \mathbf{s}^1$. Indeed, the $(j+1)$-th element of $\mathbf{s}^1$ is 0 while the $(j+1)$-th element of $\mathbf{q}^1$ is at least 1. Finally, if $j + 1 \notin I$ because $j + 1 = v$, then one can argue similarly to above that $q_j = q_{j+1} = s_j - 1 = s_{j+1} + 1$ and $\mathbf{s}^1 \neq \mathbf{q}^1$. This completes the proof of the lemma. ∎

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Without loss of generality, we can assume that $s_1 \geq \cdots \geq s_v \geq 0$. Let $c = \Sigma(\mathbf{s}) = s_1 + \cdots + s_v$ be the number of cards. Assume that $m \geq 1$ for otherwise there is nothing to do. The proof uses induction on c. The base case $c = 1$ is trivial, so assume $c \geq 2$. If $s_v = 0$, then the claim is trivially true, so assume that $s_v \geq 1$, that is, each s_i is positive.

Suppose that we have some Strategy A. Let $a_m(\mathbf{q})$ be the probability that Strategy A wins the m-round game on the $\mathbf{q}$-deck. Suppose that A selects Value j during the first step. If some value $h \in [v] \setminus \{j\}$ turns up in the first round, then Strategy A has to deal with the $(m-1)$-round game on $\mathbf{s}^h$. Let $a_{m-1}(\mathbf{s}^h)$ be the probability A that wins, when we condition on Value h appearing in Round 1. Similarly to (3), we have

$$a_m(\mathbf{s}) = \frac{1}{c} \sum_{h \in [v] \setminus \{j\}} s_h a_{m-1}(\mathbf{s}^h) \leq \frac{1}{c} \sum_{h \in [v] \setminus \{j\}} s_h g_{m-1}(\mathbf{s}^h), \qquad (13)$$

where the last inequality is obtained by applying, for each $h \neq j$, the induction assumption to the deck obtained after the removal of a card of Value h. By (3), in order to prove the optimality of the greedy strategy it is enough to prove the following statement which involves the function g_{m-1} only:

$$\frac{1}{c} \sum_{h \in [v] \setminus \{j\}} s_h g_{m-1}(\mathbf{s}^h) \leq \frac{1}{c} \sum_{h=1}^{v-1} s_h g_{m-1}(\mathbf{s}^h). \qquad (14)$$

Cancellations show that (14) is equivalent to $s_v g_{m-1}(\mathbf{s}^v) \leq s_j g_{m-1}(\mathbf{s}^j)$, which can be rewritten as

$$P(\mathbf{s}^j, \mathbf{s}^v) g_{m-1}(\mathbf{s}^v) \leq P(\mathbf{s}^v, \mathbf{s}^j) g_{m-1}(\mathbf{s}^j). \tag{15}$$

This follows from Lemma 9 by noting that $\mathbf{s}^v \succeq \mathbf{s}^j$.

Finally, suppose that the above Strategy A achieves this bound and we are not in the trivial base case $s_v = 0$. Then the inequality (13) is equality. Since each s_i is positive, we have $a_{m-1}(\mathbf{s}^h) = g_{m-1}(\mathbf{s}^h)$ for every $h \in [v] \setminus \{j\}$. The induction assumption implies that Strategy A plays greedily after the first step. Also, we must have equality in (15). Since $v \geq 3$, the second part of Lemma 9 implies that $\mathbf{s}^j = \mathbf{s}^v$. Thus $\mathbf{s}^j$ contains $s_v - 1$, which is strictly smaller than any element of $\mathbf{s}$. It follows that $s_j = s_v$. We conclude that Strategy A is the greedy strategy. ∎

2.3 Worst Adaptive Strategies

On the other hand, the case when the player wants to minimize the probability of winning, is easy.

Proof of Theorem 3. If the current position is decided, then by naming the most frequent remaining value, say 1, the player can ensure that either he loses in the next round (if Value 1 appears) or the new position is decided (because $\max(s'_1, \dots, s'_v) = s'_1$ does not change so both m' and $\Sigma(\mathbf{s}') - \max(s'_1, \dots, s'_v)$ decrease by 1). On the other hand, if a position is undecided and the game continues, then the remaining deck has cards of at least two different values. So the player survives the next round with positive probability, in which case the new position is necessarily undecided. These observations clearly imply the first part of Theorem 3.

So, suppose that $m \leq c - s_1$. Let $\mathbf{b}' = (m, 0, \dots, 0)$. Let E_i (resp. E'_i) be the event that the player's strategy (resp. the advance $\mathbf{b}'$-bid) survives the first $i \leq m$ rounds. We show by induction on i that $\mathbf{Pr}(E_i) \geq \mathbf{Pr}(E'_i)$ with the case $i = 0$ being trivially true. Let us prove the claim for $i + 1$ from the induction assumption for i. We have

$$\mathbf{Pr}(E'_{i+1}) = \mathbf{Pr}(E'_i)\, \mathbf{Pr}(E'_{i+1} \mid E'_i) = \mathbf{Pr}(E'_i)\, \frac{c - i - s_1}{c - i}.$$

On the other hand, out of $c - i$ remaining cards there are at most s_1 cards of the value mentioned by the current bid. Hence

$$\mathbf{Pr}(E_{i+1}) \geq \mathbf{Pr}(E_i)\, \frac{c - i - s_1}{c - i}. \tag{16}$$

We conclude that $\mathbf{Pr}(E_{i+1}) \geq \mathbf{Pr}(E'_{i+1})$, as required.

Finally, if some strategy deviates from the anti-greedy one, let us say this can happen in Round $i+1$ for the first time, then (16) is clearly strict (note that $\mathbf{Pr}(E_i) = \mathbf{Pr}(E'_i)$ is positive) and this strategy cannot be optimal. ∎

2.4 Playing Until All Cards Are Turned Face Up

For $\mathbf{s} \in \mathcal{V}_{v,c}$, let $g(\mathbf{s})$ denote $g_c(\mathbf{s})$, the probability that the greedy strategy wins in the game when the number of rounds equals the total number of cards. We feel that that is is the most interesting case. So, in this section, we study the properties of this function only.

v	2	3	4	5	6	7	8
$g(\mathbf{s})$	0.0142	0.0475	0.0821	0.1137	0.1416	0.1664	0.1884

v	9	10	11	12	13	14	15
$g(\mathbf{s})$	0.2080	0.2258	0.2419	0.2566	0.2701	0.2826	0.2942

Table 1: The values of $g(4,\ldots,4)$

Table 1 lists the value of $g(\mathbf{s})$ rounded down to the 4-th decimal digit, where $\mathbf{s} = (4,\ldots,4)$ is the regular vector of length $v \leq 15$. By looking at the values of $g(4,\ldots,4)$ one notices that this is an increasing function of v. In fact, the following more general phenomenon happens.

Proposition 10 *Let $v \geq 1$ and let $\mathbf{s}$ be a v-sequence of non-negative integers. Let $\mathbf{q}$ be obtained from $\mathbf{s} \in \mathcal{V}_{v,c}$ by inserting an extra term s_{v+1}. (For convenience, we do not require that the sequences are monotone; in particular, the inserted element s_{v+1} need not be the smallest element of $\mathbf{q}$.)*

Then $g(\mathbf{q}) \geq g(\mathbf{s})$. Moreover, if all elements of $\mathbf{s}$ are positive, then this inequality is strict.

Proof. If $s_i = 0$ for some $i \in [v]$, then the claimed inequality $g(\mathbf{q}) \geq g(\mathbf{s})$ is trivially true since both parts equal 1. So suppose otherwise. By Theorem 2 it is enough to give an example of a strategy which wins on the $\mathbf{q}$-deck with probability strictly larger than $g(\mathbf{s})$.

The player plays in the following manner. If no cards of Value $v+1$ remain in the deck, then Player wins by naming Value $v+1$. Otherwise, he ignores Value $v+1$ and applies the greedy strategy with respect to Values $1, \ldots, v$. In other words, he mentions a least frequent remaining value among $1, \ldots, v$ unless there is a sure win by naming Value $v+1$.

Clearly, had the player completely ignored Value $v+1$, his chances of winning would have been exactly $g(\mathbf{s})$. However, with positive (although perhaps very small) probability all cards of Value $v+1$ come on the top of the shuffled deck. This is a win for the player, which pushes his overall chance strictly above $g(\mathbf{s})$. ∎

Here is another "monotonicity" property of the function $g(\mathbf{s})$.

Proposition 11 *Let* $\mathbf{s} = (s_1, \ldots, s_v)$ *be an arbitrary (not necessarily monotone) sequence and let* $\mathbf{q} = (s_1{+}1, s_2, \ldots, s_v)$*. Then* $g(\mathbf{s}) \geq g(\mathbf{q})$*. Moreover, if* $s_i > 0$ *for every* $2 \leq i \leq v$*, then the inequality is strict.*

Proof. In order to prove the inequality, it is enough by Theorem 2 to specify a strategy for the $\mathbf{s}$-deck whose probability of winning is at least $g(\mathbf{q})$. A randomized strategy will also do here.

The player takes a uniformly shuffled $\mathbf{s}$-deck and inserts randomly a new card, the *joker*, with all $\Sigma(\mathbf{s})+1$ positions being equally likely. Then he uses the greedy strategy, regarding the joker as a card of Value 1. Also, we may agree that if 1 is among the least frequent remaining values, then the player necessarily names 1.

If the joker would cause a coincidence as a regular card of Value 1, then the player would win with probability exactly $g(\mathbf{q})$. But let the joker be a lucky card and never give a coincidence. Thus, effectively, the player plays against the $\mathbf{s}$-deck. The probability of win (if the player follows the same strategy) cannot go down. This proves the desired inequality.

Moreover, the inequality is strict if s_i is positive for each $2 \leq i \leq v$. Indeed, it is possible to order the $\mathbf{q}$-deck so that the greedy strategy loses, but it would have won if one of the Value 1 cards were replaced by the joker. This is done by putting some cards of Value 1 on the top of the deck so that the greedy strategy will survive up until the first time it has to name Value 1, then placing a Value 1 card at that spot, and then again ensuring that the greedy strategy would survive the remainder of the deck if that card were replaced by the joker. ∎

Also, the following more general theorem implies that the entries in the second row of Table 1 converge to 1.

Theorem 12 *For every integer ℓ and every real $\varepsilon > 0$ there is a v_0 such that $g(\mathbf{s}) \geq 1 - \varepsilon$ for every deck $\mathbf{s} = (s_1, \ldots, s_v)$ with $v \geq v_0$ and each s_i being at most ℓ.*

Proof. Fix ℓ and $\varepsilon > 0$, and let $v \to \infty$. Let $\mathbf{s}$ satisfy the assumptions of the theorem. Assume that each s_i is positive for otherwise $g(\mathbf{s}) = 1$ and there is nothing to do. Let $c = \Sigma(\mathbf{s}) \geq v$. By Theorem 2 it is enough to specify a strategy that wins with probability at least $1 - \varepsilon$.

Let $a = \lfloor v/\ln v \rfloor$, where ln denotes the natural logarithm. (We do not try to optimize the values.) By the pigeonhole principle, we can find a number $m \in [\ell]$ and a set $M \subseteq [v]$ such that $|M| = \lceil v/\ell \rceil$ and $s_i = m$ for every $i \in M$. Let us call the values in M *special* and the remaining ones *ordinary*. Let the player name ordinary values in an arbitrary fashion until the deck runs out of some special value in which case the player starts naming this value (and necessarily wins).

The probability that the player loses at any particular round $i \leq a$ is at most $\ell/(c - a + 1) \leq \ell/(v - a + 1)$ whatever the player does. By the union bound, the probability that the player loses within the first a rounds is at most $a \times \ell/(v - a + 1) \leq \varepsilon/2$.

For $i \in M$, let X_i be the event that all cards of Value i appears among the first a cards in a uniformly shuffled $\mathbf{s}$-deck. Let the random variable N be the number of indices $i \in M$ such that X_i occurs. In order to prove the theorem, it is enough to show that

$$\mathbf{Pr}(N = 0) \leq \varepsilon/2. \tag{17}$$

We use the second moment method (see, for example, Alon and Spencer [1, Chapter 4]) to prove (17). Recall that ℓ is fixed, $1 \leq m \leq \ell$, and $v \to \infty$. Thus $a \to \infty$ and $a/c \to 0$.

The probability $\mathbf{Pr}(X_i) = \binom{a}{m}\binom{c}{m}^{-1}$ does not depend on $i \in M$; denote it by p. Since $c > a$, the expectation of N is

$$\mathbf{E}(N) = |M|p \geq \frac{v}{\ell} \times \left(\frac{a - m + 1}{c - m + 1}\right)^m \to \infty.$$

Also, the covariance of X_i and X_j for distinct $i, j \in M$ is

$$\mathbf{Cov}(X_i, X_j) = \mathbf{Pr}(X_j \wedge X_j) - p^2 = \frac{\binom{a}{m}\binom{a-m}{m}}{\binom{c}{m}\binom{c-m}{m}} - \frac{\binom{a}{m}^2}{\binom{c}{m}^2} = o(p^2).$$

Thus $\mathbf{Var}(N) \leq \mathbf{E}(N) + \sum_{i \neq j} \mathbf{Cov}(X_i, X_j) = o(\mathbf{E}(N)^2)$. By Chebyshev's inequality ([1, Theorem 4.3.1]), the probability that $N = 0$ is at most

$\mathbf{Var}(N)/\mathbf{E}(N)^2 = o(1)$. In particular, (17) holds if v is sufficiently large, depending only on ℓ and ε. ∎

Unfortunately, we could not find any closed formula for $g(\mathbf{s})$. But for some special cases, explicit formulas exists. One example is

$$g(q,k,1) = \frac{1}{q+1} + \frac{1}{k+1} - \frac{1}{q+k+1}. \tag{18}$$

Here is a direct combinatorial proof of (18). Suppose we have one Ace, $q \geq 1$ Queens, and $k \geq 1$ Kings. The greedy strategy keeps calling Ace until either the Ace appears (and the player loses) or Queens or Kings run out (and the player wins). The probability that the Ace comes after all Queens is $1/(q+1)$, after all Kings is $1/(k+1)$, after all Kings and Queens is $1/(q+k+1)$. A simple inclusion-exclusion gives (18).

Also we have, for example,

$$\begin{aligned}
g(i,2,2) &= \frac{1}{6} + \frac{8}{3(i+1)} - \frac{6}{i+2} + \frac{6}{i+3} - \frac{8}{3(i+4)}, \quad i \geq 2,\\
g(i,3,2) &= \frac{1}{10} + \frac{2}{i+1} - \frac{9}{i+3} + \frac{12}{i+4} - \frac{5}{i+5}, \quad i \geq 3,\\
g(i,4,2) &= \frac{1}{15} + \frac{2}{i+1} - \frac{2}{i+2} + \frac{4}{i+3} - \frac{16}{i+4} + \frac{20}{i+5} - \frac{8}{i+6}, \quad i \geq 4,\\
g(i,3,3) &= \frac{1}{20} + \frac{51}{10\ (i+1)} - \frac{39}{2\ (i+2)}\\
&+ \frac{39}{i+3} - \frac{48}{i+4} + \frac{33}{i+5} - \frac{48}{5\ (i+6)}, \quad i \geq 3.
\end{aligned}$$

Each of the above identities can be verified by induction on i using (3) (and the previous identities). The calculations are straightforward but messy, so we omit them. Further identities along these lines can be written but we could not spot any pattern. We decomposed the right-hand sides into partial fractions as this representation looked most aesthetically pleasing. We do not have any interpretation of the coefficients except for the constant terms: namely, $\frac{1}{6} = g(2,2)$, $\frac{1}{10} = g(3,2)$, $\frac{1}{15} = g(4,2)$, and $\frac{1}{20} = g(3,3)$. This makes sense because, for any fixed $s_2, \ldots, s_v$, we have

$$\lim_{s_1 \to \infty} g(s_1, s_2, \ldots, s_v) = g(s_2, \ldots, s_v).$$

This identity can be proved by noting that the probability that the last $l = \max(s_2, \ldots, s_v) + 1$ cards of a uniformly shuffled deck will all have Value 1 is $1 - o(1)$ as $s_1 \to \infty$. (Indeed, the expected number of cards with value different from 1 among the last l cards is $l \times \sum_{i=2}^{v} \frac{s_i}{s_1 + \cdots + s_v} = o(1)$

so by Markov's inequality there is none almost surely.) Thus, if the above event happens, then the greedy strategy never names Value 1. Hence, it wins with probability $g(s_2, \dots, s_v) + o(1)$.

3 Advance Strategies

Recall that, for vectors $\mathbf{b}$ and $\mathbf{s}$ of the same length v with $\Sigma(\mathbf{b}) \le \Sigma(\mathbf{s})$, $\mathbf{Pr}(\mathbf{b}, \mathbf{s})$ denotes the probability that the advance bid $\mathbf{b}$ wins against the $\mathbf{s}$-deck. Also, we call $\mathbf{b}$ an *optimal bid* for the $\mathbf{s}$-deck if $\mathbf{Pr}(\mathbf{b}', \mathbf{s}) \le \mathbf{Pr}(\mathbf{b}, \mathbf{s})$ for every v-vector $\mathbf{b}'$ with $\Sigma(\mathbf{b}') = \Sigma(\mathbf{b})$.

Here we prove Theorem 5. For this purpose, it will be convenient to prove a weaker version of it first, namely that *at least one* optimal bid is almost regular. This clearly follows from Lemma 13 below. Although the conclusion of Lemma 13 that $s_i > s_j$ implies $b_i \le b_j$ is not needed for the proof of Theorem 5, we include it here since this makes the proof of Lemma 13 only slightly longer.

Lemma 13 *For every composition vector $\mathbf{s} = (s_1, \dots, s_v)$ and any integer $m \le \Sigma(\mathbf{s})$ there is an optimal advance bid $\mathbf{b}$ with $\Sigma(\mathbf{b}) = m$ such that, for every $i, j \in [v]$, $s_i = s_j$ implies that $|b_i - b_j| \le 1$ and $s_i > s_j$ implies that $b_i \le b_j$.*

Proof. The lemma is trivial if some s_i is 0 or if $v = 1$. Also, the lemma follows from Proposition 1 if $v = 2$. So assume otherwise. Among all optimal advance bids $\mathbf{b}$ with $\Sigma(\mathbf{b}) = m$ choose one that minimizes

$$\sum_{1 \le i < j \le v} |(s_i + b_i) - (s_j + b_j)| . \tag{19}$$

We claim that this vector $\mathbf{b}$ satisfies the lemma. Suppose on the contrary that this is not the case. Without loss of generality we can assume that the conclusion of the lemma is violated for indices 1 and 2 with $b_1 > b_2$. Thus we have that $s_1 = s_2$ and $b_1 \ge b_2 + 2$ or that $s_1 > s_2$. In either case, we have $s_1 + b_1 \ge s_2 + b_2 + 2$.

Let $\mathbf{b}' = (b'_1, \dots, b'_v)$, where $b'_1 = b_1 - 1 \ge 0$, $b'_2 = b_2 + 1$, and $b'_i = b_i$ for $i \ge 3$. Thus $\mathbf{b}'$ is obtained from $\mathbf{b}$ by replacing one guess of Value 1 by Value 2. It is easy to see that for any numbers $a \ge b + 2$ and c, we have $|a-c|+|b-c| \ge |(a-1)-c|+|(b+1)-c|$ while clearly $|a-b| > |(a-1)-(b+1)|$. This observation, when applied to $a = s_1 + b_1$, $b = s_2 + b_2$, and $c = s_i + b_i$

for $3 \le i \le v$, shows that the replacement of $\mathbf{b}$ by $\mathbf{b}'$ would strictly decrease the expression in (19). Hence, $\mathbf{b}'$ cannot be optimal, that is,

$$\mathbf{Pr}(\mathbf{b}', \mathbf{s}) < \mathbf{Pr}(\mathbf{b}, \mathbf{s}). \tag{20}$$

Let us set up some notation that we need in order to derive a contradiction from (20). Let $c = \Sigma(\mathbf{s})$ be the total number of cards and recall that $m = \Sigma(\mathbf{b})$. Let us order both bids $\mathbf{b}$ and $\mathbf{b}'$ by value and let B_i (resp. B_i') consist of the positions where the bid $\mathbf{b}$ (resp. $\mathbf{b}'$) suggests Value i. Thus the sets B_i (as well as the sets B_i') partition $[m]$ and, for every $i \in [v]$, we have $|B_i| = b_i$ and $|B_i'| = b_i'$. Also, $B_i = B_i'$ for every $i \in [3, v]$ while $B_1 = [b_1] = B_1' \cup \{b_1\}$, and $B_2' = [b_1, b_1 + b_2] = B_2 \cup \{b_1\}$.

Let C be the set of all cards in the deck. Let $S_i \subseteq C$ consist of all cards of Value i. A random shuffling of the deck is encoded by a bijection $\sigma : C \to [c]$. (For convenience, assume that $C \cap [c] = \emptyset$.) The value $\sigma(x)$ is the position at which Card x appears. Thus, for example, the bid $\mathbf{b}$ wins for σ if and only if $B_i \cap \sigma(S_i) = \emptyset$ for every $i \in [v]$. Such a bijection σ will be called a $\mathbf{b}$-*winning bijection*. Of course, only the first m card values, namely $\sigma^{-1}(1), \ldots, \sigma^{-1}(m)$, are needed to determine the outcome of the game but we record the whole bijection σ for the convenience of calculations.

The bijection σ is chosen uniformly at random from all $c!$ choices. We will need the following random variables determined by σ. Let $D \in [v]$ be the value of the card that appears in Position b_1. (Recall that b_1 is the unique element of $B_1 \setminus B_1'$.) Let $N_1 = |B_1' \cap \sigma(S_2)|$ and $N_2 = |B_2 \cap \sigma(S_1)|$.

Let Φ consist of all bijections $\sigma : C \to [c]$ that produce different outcomes for the bids $\mathbf{b}$ and $\mathbf{b}'$, that is, those for which one bid wins while the other loses. Formally,

$$\Phi = \{\sigma : D \in \{1, 2\},\ \sigma(S_1) \cap B_1' = \emptyset,\ \forall\, i \in [2, v]\ \sigma(S_i) \cap B_i = \emptyset\}.$$

By definition, any bijection not in Φ contributes the same amount to both sides of (20). Hence, (20) implies that $\Phi \neq \emptyset$ (so we can condition on Φ) and that we have the following inequality between the conditional probabilities:

$$\mathbf{Pr}(D = 1 \,|\, \sigma \in \Phi) < \mathbf{Pr}(D = 2 \,|\, \sigma \in \Phi). \tag{21}$$

Let $W = \{N_1 + N_2 : \sigma \in \Phi\}$. Fix an arbitrary $w \in W$. Let

$$\Phi_w = \{\sigma \in \Phi : N_1 + N_2 = w\}.$$

Since $w \in W$, the set Φ_w is non-empty. We have

$$\mathbf{Pr}(D = 1 \,|\, \sigma \in \Phi_w) \;=\; \sum_{i=0}^{w} \frac{s_1 - w + i}{s_1 + s_2 - w}\, \mathbf{Pr}(N_1 = i \,|\, \sigma \in \Phi_w), \quad (22)$$

$$\mathbf{Pr}(D = 2 \,|\, \sigma \in \Phi_w) \;=\; \sum_{i=0}^{w} \frac{s_2 - i}{s_1 + s_2 - w}\, \mathbf{Pr}(N_1 = i \,|\, \sigma \in \Phi_w). \quad (23)$$

Note that $w < s_1 + s_2$ because $w \in W$ implies that at least $w + 1$ cards of Values 1 or 2 are present in the deck. If we subtract (23) from (22) and multiply the result by $s_1 + s_2 - w \geq 1$, we get by (21) that

$$\sum_{i=0}^{w} (2i + s_1 - s_2 - w)\, \mathbf{Pr}(N_1 = i \,|\, \sigma \in \Phi_w) = \mathbf{E}(2N_1 + s_1 - s_2 - w \,|\, \sigma \in \Phi_w) < 0, \quad (24)$$

which is the conditional expectation of $2N_1 + s_1 - s_2 - w = N_1 - N_2 + s_1 - s_2$. Let us establish a contradiction by showing that it is non-negative.

Trivially, each bijection $\sigma \in \Phi_w$ with $N_2 < s_1 - s_2$ makes a positive contribution to the left-hand side of (24). Let us consider the remaining cases. Define

$$U_w = \{(N_1, N_2 - s_1 + s_2) : \sigma \in \Phi_w\}.$$

Claim 1 If $(l, k) \in U_w$ and $k > l$, then $(k, l) \in U_w$.

Proof of Claim. We show by induction on i that for every $i = 0, \ldots, k - l$, we have $(l+i, k-i) \in U_w$. Suppose this is true for some i with $0 \leq i < k-l$. Take a witness $\sigma \in \Phi_w$. Pick an $x \in B_1' \setminus \sigma(S_2)$. This set is non-empty because $|B_1' \cap \sigma(S_2)| = l + i < k$ while $(l, k) \in U_w$ implies $|B_1'| = b_1' \geq b_2 \geq k + s_1 - s_2 \geq k$. Next, pick an element $y \in B_2 \cap \sigma(S_1)$, this set having $k - i + s_1 - s_2 > 0$ elements. Also, pick an element $z \in \sigma(S_2) \setminus B_1'$. This set is non-empty because $(l, k) \in U_w$ implies that $k + s_1 - s_2 \leq s_2 \leq s_1$, that is, $k \leq s_2$, while $|B_1' \cap \sigma(S_2)| = l + i < k \leq s_2$. (Note that we allow z to be b_1.) Let a new bijection σ' be obtained by composing σ with the permutation of $[c]$ that fixes every element of $[c]$ except it permutes x, y, z cyclically in this order. Then $\sigma' \in \Phi_w$, which shows that $(l + i + 1, k - i - 1) \in U_w$. This finishes the inductive proof. ▌

For $(k, l) \in U_w$, let

$$\Phi_{k,l} = \{\sigma \in \Phi : N_1 = k,\ N_2 = l + s_1 - s_2\} \neq \emptyset.$$

Clearly, the sets $\Phi_{k,l}$ are pairwise disjoint and their union over all $(k, l) \in U_w$ is exactly Φ_w. By definition, for every $(k, l) \in U_w$ we have

$$k + l = w - s_1 + s_2. \quad (25)$$

If $(k,l) \in U_w$, but $(l,k) \notin U_w$, then $k > l$ by Claim 1. By (25), we have $k > (w - s_1 + s_2)/2$. Thus for an arbitrary $\sigma \in \Phi_{k,l}$, we have $N_1 > (w - s_1 + s_2)/2$. Here, the contribution to the left-hand side of (24) is strictly positive.

Thus, let us consider the contribution to (24) by a pair of numbers $k \geq l$ such that $k + l = w - s_1 + s_2$ and both (k,l) and (l,k) belong to U_w. Let us prove that

$$|\Phi_{k,l}| \geq |\Phi_{l,k}|. \tag{26}$$

Let us calculate $|\Phi_{k,l}|$. First, we have to map some k elements of S_2 into B_1' (giving $\binom{s_2}{k}\binom{b_1'}{k}k!$ possibilities). Then we have map $l+s_1-s_2$ elements of S_1 into B_2 (giving $\binom{s_1}{l+s_1-s_2}\binom{b_2}{l+s_1-s_2}(l+s_1-s_2)!$ ways). Finally, we have to take care of the remaining unassigned cards that include $s_1-(l+s_1-s_2) = s_2 - l$ cards of Value 1, $s_2 - k$ cards of Value 2, and s_i cards of Value i for $i \geq 3$. The number M of possibilities at this step does not depend on the previous choices. Hence

$$|\Phi_{k,l}| = \binom{s_2}{k}\binom{b_1'}{k}k! \times \binom{s_1}{l+s_1-s_2}\binom{b_2}{l+s_1-s_2}(l+s_1-s_2)! \times M.$$

Similarly, we obtain

$$|\Phi_{l,k}| = \binom{s_2}{l}\binom{b_1'}{l}l! \times \binom{s_1}{k+s_1-s_2}\binom{b_2}{k+s_1-s_2}(k+s_1-s_2)! \times M'.$$

Note that the only difference in the definition of M' when compared to that of M is that we have s_2-k cards of Value 1 and s_2-l cards of Value 2. But the cards of Value 1 and 2 behave identically in the definition of M or M', so every legitimate M-assignment gives a legitimate M'-assignment by swapping Values 1 and 2. Hence, $M = M'$. Also, since $\Phi_{k,l}$ and $\Phi_{l,k}$ are non-empty, we have $M = M' > 0$. Thus we have

$$\begin{aligned}\frac{|\Phi_{l,k}|}{|\Phi_{k,l}|} &= \frac{k!(l+s_1-s_2)!}{l!(k+s_1-s_2)!} \times \frac{(b_1'-k)!(b_2-l-s_1+s_2)!}{(b_1'-l)!(b_2-k-s_1+s_2)!}\\ &= \prod_{i=0}^{k-l-1}\frac{k-i}{k+s_1-s_2-i} \times \prod_{j=0}^{k-l-1}\frac{b_2-l-s_1+s_2-j}{b_1'-l-j} \ \leq\ 1.\end{aligned}$$

Here we used the inequalities $s_1 \geq s_2$ and $b_1' \geq b_2 \geq k \geq l \geq 0$. (Note that, since $(l,k) \in U_w$, we have $b_2 \geq k + s_1 - s_2 \geq k$.) This proves (26).

Now, $k \geq l$ implies by (25) that $2k + s_1 - s_2 - w \geq 0 \geq 2l + s_1 - s_2 - w$.

By (26), the contribution of $\Phi_{k,l} \cup \Phi_{l,k}$ to the left-hand side of (24) is

$$\begin{aligned}&(2k + s_1 - s_2 - w)\,\frac{|\Phi_{k,l}|}{|\Phi_w|} + (2l + s_1 - s_2 - w)\,\frac{|\Phi_{l,k}|}{|\Phi_w|}\\ &\qquad\geq\ \frac{|\Phi_{k,l}| + |\Phi_{l,k}|}{2|\Phi_w|}\,(2k + s_1 - s_2 - w + 2l + s_1 - s_2 - w) = 0.\end{aligned}$$

Putting all together, we obtain a contradiction to (24), proving the lemma. ∎

Proof of Theorem 5. Suppose on the contrary that the theorem is false, that is, we can find an optimal vector that is not almost regular. By iteratively changing its entries as in the proof of Lemma 13, we eventually reach an almost regular optimal vector $\mathbf{b}'$. Let $\mathbf{b}$ be the optimal bid from the previous step, i.e., the last bid that contradicts Theorem 5 from the obtained chain of optimal bids. Without loss of generality, assume that $b_1' = b_1 - 1$ and $b_2' = b_2 + 1$. Let us recycle the notation that we used in the proof of Lemma 13. Let $U = \cup_{w\in W} U_w$.

Since $b_i' \leq s$ for each $i \in [v]$, we can find a partition $[c] = \cup_{i=1}^{v} C_i$ such that $|C_i| = s$ and $C_i \supseteq B_i'$ for every $i \in [v]$. Note that $b_1 \in B_2' \subseteq C_2$.

A bijection σ that maps bijectively each S_i into C_{i+1}, where we agree that $C_{v+1} = C_1$, shows that $\Phi \neq \emptyset$ and that $(0, b_2) \in U$. (Recall that $v \geq 3$ by the assumption of the theorem.) Thus one can condition on the non-empty set Φ. It follows that each of inequalities (20), (21), and (24) is equality now. We must have $b_2 = 0$ for otherwise the inequality (26) is a strict for $(k, l) = (b_2, 0)$. (Note that $(b_2, 0), (0, b_2) \in U$ by Claim 1 of Lemma 13.) Also, we have $b_1 \geq 2$ for otherwise $\mathbf{b}$ is almost regular, contradicting our assumption.

We cannot have $(k, 0) \in U$ with some $k > 0$ (for this would make (26) strict if $(0, k) \in U$ or would directly make (24) strict otherwise). It follows that $v \leq 3$: otherwise a bijection $\sigma : C \to [c]$ that maps S_i into C_{i+1} for $i \in [3, v-1]$ and satisfies $\sigma(S_1) = C_3$, $\sigma(S_2) = C_1$ and $\sigma(S_v) = C_2$ shows that $(b_1 - 1, 0) \in U$, a contradiction. But if $v = 3$ and $s \geq 2$, then we get a contradiction $(1, 0) \in U$ by taking σ that maps some element from each of S_1, S_2, and S_3 into correspondingly C_3, C_1, and $C_2 \setminus \{b_1\}$, and then maps the remainder of each S_i into the unassigned part of C_{i+1} for $i \in [3]$. Finally, the case $v = 3$ and $s = 1$ (and $1 \leq m \leq 3$) is easily seen to satisfy Theorem 5. ∎

Table 2 lists all optimal advance bids $\mathbf{b}$ with $\Sigma(\mathbf{b}) = \Sigma(\mathbf{s})$ for all 3-vectors $\mathbf{s} = (s_1, s_2, s_3)$ such that $s_1 \geq s_2 \geq s_3 \geq 1$, and $\Sigma(\mathbf{s}) \leq 11$. If we

s	b
$\{1,1,1\}$	$\{0,1,2\}$
	$\{1,1,1\}$
$\{2,1,1\}$	$\{0,2,2\}$
$\{3,1,1\}$	$\{0,2,3\}$
$\{2,2,1\}$	$\{0,1,4\}$
	$\{0,2,3\}$
	$\{1,1,3\}$
$\{4,1,1\}$	$\{0,3,3\}$
$\{3,2,1\}$	$\{0,2,4\}$
$\{2,2,2\}$	$\{2,2,2\}$
$\{5,1,1\}$	$\{0,3,4\}$
$\{4,2,1\}$	$\{0,2,5\}$
$\{3,3,1\}$	$\{0,1,6\}$
	$\{0,2,5\}$
	$\{1,1,5\}$
$\{3,2,2\}$	$\{0,3,4\}$
	$\{1,3,3\}$
$\{6,1,1\}$	$\{0,4,4\}$
$\{5,2,1\}$	$\{0,2,6\}$
	$\{0,3,5\}$

s	b
$\{4,3,1\}$	$\{0,2,6\}$
$\{4,2,2\}$	$\{0,4,4\}$
$\{3,3,2\}$	$\{2,2,4\}$
$\{7,1,1\}$	$\{0,4,5\}$
$\{6,2,1\}$	$\{0,3,6\}$
$\{5,3,1\}$	$\{0,2,7\}$
$\{5,2,2\}$	$\{0,4,5\}$
$\{4,4,1\}$	$\{0,1,8\}$
	$\{0,2,7\}$
	$\{1,1,7\}$
$\{4,3,2\}$	$\{0,3,6\}$
	$\{0,4,5\}$
	$\{1,3,5\}$
$\{3,3,3\}$	$\{3,3,3\}$
$\{8,1,1\}$	$\{0,5,5\}$
$\{7,2,1\}$	$\{0,3,7\}$
$\{6,3,1\}$	$\{0,2,8\}$
$\{6,2,2\}$	$\{0,5,5\}$
$\{5,4,1\}$	$\{0,2,8\}$
$\{5,3,2\}$	$\{0,4,6\}$

s	b
$\{4,4,2\}$	$\{2,2,6\}$
$\{4,3,3\}$	$\{2,4,4\}$
$\{9,1,1\}$	$\{0,5,6\}$
$\{8,2,1\}$	$\{0,3,8\}$
	$\{0,4,7\}$
$\{7,3,1\}$	$\{0,2,9\}$
	$\{0,3,8\}$
$\{7,2,2\}$	$\{0,5,6\}$
$\{6,4,1\}$	$\{0,2,9\}$
$\{6,3,2\}$	$\{0,4,7\}$
$\{5,5,1\}$	$\{0,1,10\}$
	$\{0,2,9\}$
	$\{1,1,9\}$
$\{5,4,2\}$	$\{0,3,8\}$
	$\{0,4,7\}$
	$\{1,3,7\}$
$\{5,3,3\}$	$\{0,5,6\}$
	$\{1,5,5\}$
$\{4,4,3\}$	$\{3,3,5\}$

Table 2: Optimal advance bids $\mathbf{b}$ with $\Sigma(\mathbf{b}) = \Sigma(\mathbf{s})$ for some 3-vectors $\mathbf{s}$.

have $s_i = \cdots = s_j$ for some $i < j$, then, in order to save space, we include only those optimal $\mathbf{b}$ such that $b_i \le \cdots \le b_j$. The reader is welcome to experiment with our computer code, which can be found in [20].

By looking at Table 2 and by computing further optimal vectors, one can spot patterns in some special cases (and perhaps even rigorously prove them) but the general solution to Problem 4 (or even just a general conjecture) evaded us so far.

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