Remarks on a Paper by H. Bielak on Size Ramsey Numbers

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Abstract

The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the smallest number of edges that an (F_1, F_2)-arrowing graph can have. Let $S_{i,n}$ be obtained from the star $K_{1,n}$ by subdividing one edge by new $i-1$ vertices. Bielak [Periodica Math. Hung. 18 (1987) 27–38] showed that $\hat{r}(S_{1,n}, S_{1,n}) = 4n-2$ and that $\hat{r}(S_{2,n}, S_{2,n}) \le 5n+3$. We compute asymptotically all unknown values of $\hat{r}(S_{\mu,n}, S_{\nu,n})$, $0 \le \mu \le \nu \le 2$. In particular, we show that $\hat{r}(S_{2,n}, S_{2,n}) = \frac{19}{4}n + O(1)$.

1 Introduction

A graph G *arrows* a pair (F_1, F_2) of graphs, which is denoted by $G \to (F_1, F_2)$, if for any blue-red colouring of the edge set of G there is a blue copy of F_1 or a red copy of F_2. The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the smallest number of edges in a such graph G. This function is usually difficult to compute and often we do not know the answer even for simple pairs (F_1, F_2).

This problem was first studied in a series of papers by Burr, Erdős, Faudree, Rousseau, Schelp and others in the late 70s. The papers [6, 4, 1, 8,

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9, 3, 2, 5, 7, 11, 10, 12, 13, 14], to name a few, as well as the present paper, deal with the case when we forbid bipartite graphs.

Let $S_{i,n}$ be obtained from the star $K_{1,n}$ by replacing one edge with a path of length $i+1$. (In particular, $S_{0,n} = K_{1,n}$.) Bielak [3] showed that $\hat{r}(S_{1,n}, S_{1,n}) = 4n-2$ and that $\hat{r}(S_{2,n}, S_{2,n}) \leq 5n+3$ for $n \geq 3$.

This research initiated as an attempt to compute $\hat{r}(S_{2,n}, S_{2,n})$ which happens to be around $\frac{19}{4}n$. Then it has been realised that we can get, with little extra work, some related asymptotic results as the following theorem asserts.

Theorem 1 *For* $0 \leq \mu \leq \nu \leq 2$ *we have* $\hat{r}(S_{\mu,n}, S_{\nu,n}) = \alpha_{\mu,\nu}n + O(1)$, *where* $\alpha_{0,0} = 2$, $\alpha_{0,1} = \alpha_{0,2} = \alpha_{1,1} = 4$, $\alpha_{1,2} = \frac{14}{3}$, *and* $\alpha_{2,2} = \frac{19}{4}$.

The case $\mu = \nu = 0$ is trivial while the case $\mu = \nu = 1$ is done by Bielak [3]. The rest of this paper is dedicated to settling the remaining cases. We do not go for the exact computation as this would considerably increase the size of this article. (The proof that $\hat{r}(S_{1,n}, S_{1,n}) = 4n-2$ occupies more than five pages in [3].)

2 Upper Bounds

We have to prove the upper bound on $\alpha_{0,2}$, $\alpha_{1,2}$ and $\alpha_{2,2}$ only.

Lemma 2 $\hat{r}(S_{0,n}, S_{2,n}) \leq 4n-1$, $n \geq 1$.

Proof. Let G be obtained from $K_{2,2n}$ with parts $V_1 \cup V_2$, $V_1 = \{x_1, x_2\}$, by removing one edge, incident to x_2, say. For any colouring of G without a blue $S_{0,n}$, we have $d_{\text{red}}(x_1) \geq n+1$ and $d_{\text{red}}(x_2) \geq n$, where, for example, $d_{\text{red}}(x_1) = |\Gamma_{\text{red}}(x_1)|$ denotes the number vertices sending a red edge to x_1. Thus, $\Gamma_{\text{red}}(x_1) \cap \Gamma_{\text{red}}(x_2) \neq \emptyset$ and we have a red $S_{2,n}$, as required. ∎

Lemma 3 $\hat{r}(S_{1,n}, S_{2,n}) \leq 5n - \lfloor\frac{n-1}{3}\rfloor + 7$, $n \geq 1$.

Proof. Take $K_{2,2n+3}$ on $V_1 \cup V_2$. Insert into V_2 a graph T formed from vertex-disjoint copies of $K_{1,2}$ plus at most one copy of $K_{1,3}$ or $K_{1,4}$ so that

$v(T) = n+2$. The graph $G = K_{2,2n+3} \cup T$ has the stated size. Suppose that we can find an $(S_{1,n}, S_{2,n})$-free colouring of G. Let $V_1 = \{x_1, x_2\}$.

If there is $y \in V_2$ such that both $\{x_1, y\}$ and $\{x_2, y\}$ are blue, then $d_{\text{blue}}(x_i) \leq n-1$, that is, $d_{\text{red}}(x_i) \geq n+4$, $i = 1,2$. In particular, $\Gamma_{\text{red}}(x_1) \cap \Gamma_{\text{red}}(x_2) \neq \emptyset$, which gives us a red $S_{2,n}$, a contradiction.

Next, suppose that there is $z \in V_2$ such that both $\{x_1, z\}$ and $\{x_2, z\}$ are red. If $d_{\text{red}}(x_1) = 1$, then $G[x_2, V_2]$ is red. (For $A, B \subset V(G)$, $G[A, B]$ consists of all edges of G connecting A to B.) Now, T contains a blue edge or a red $K_{1,2}$, either case leading to a contradiction. Thus, $d_{\text{red}}(x_1) \geq 2$ and, likewise, $d_{\text{red}}(x_2) \geq 2$. To avoid a red $S_{2,n}$, we must have $d_{\text{red}}(x_1), d_{\text{red}}(x_2) \leq n$, which gives us $y \in V_2$ sending two blue edges to V_1, which is, as we already know, impossible.

Let $A \subset V_2$ consist of those vertices that send a red edge to x_1. We know that $G[x_2, A]$ is blue and $G[x_2, B]$ is red, where $B = V_2 \setminus A$. Up to symmetry, we can assume that $|A| \geq |B|$. Then $|A| \geq n+2$ and A intersects some $K_{1,2} \subset T$. Trivial considerations show that there is no feasible way to colour this $K_{1,2}$, which finishes the proof. ∎

Lemma 4 *$\hat{r}(S_{2,n}) \leq 5n - \lfloor \frac{n+2}{4} \rfloor + 8$, $n \geq 2$.*

Proof. Take $K_{2,2n+3}$ and add, to the bigger part V_2, a graph T made of vertex-disjoint $K_{1,3}$'s plus at most one of $K_{1,4}$, $K_{1,5}$, or $K_{1,6}$, so that $v(T) = n+2$. We have the required number of edges. Suppose that there is a colouring without a monochromatic $S_{2,n}$. Let $V_1 = \{x_1, x_2\}$.

Suppose that, say, the blue degree of x_1 is at most 1. Clearly, x_2 can send at most one red edge to $\Gamma_{\text{red}}(x_1)$. Some two edges of a $K_{1,3} \subset T$ have the same colour, which gives us a monochromatic $S_{2,n}$, a contradiction. Thus we can assume that x_i is incident to at least 2 edges of each colour, $i = 1,2$. Also, like in Lemma 3, one can show that for any $y \in V_2$ the edges $\{x_1, y\}$ and $\{x_2, y\}$ have different colours. Assume, by symmetry, that $d_{\text{red}}(x_1) \geq d_{\text{blue}}(x_1)$. Now, $V(T) \cap \Gamma_{\text{red}}(x_1) \neq \emptyset$ but there is no feasible way to colour a $K_{1,3}$ intersecting $\Gamma_{\text{red}}(x_1)$. ∎

3 Lower Bounds

We have to prove the lower bound on $\alpha_{\mu,\nu}$ for $(\mu,\nu) = (0,1)$, $(1,2)$, and $(2,2)$ only. Let n be large. First, we state a few lemmas applicable for any of these pairs (μ,ν). Let $G \to (S_{\mu,n}, S_{\nu,n})$ be minimum. Let

$$H = \{x \in V(G) : d(x) \geq n\},$$

$\overline{H} = V(G) \setminus H$, $h = |H|$, and $H = \{x_1, \ldots, x_h\}$ with $d(x_1) \geq \cdots \geq d(x_h)$.

As $e(G) < 5n - \binom{5}{2}$, we conclude that $h \leq 4$.

Lemma 5 $\Delta(G) \geq 2n - 1$.

Proof. Otherwise, we can find a $K_{1,n}$-free colouring of G by extending, for $i = 1, \ldots, h$, such a colouring from $G_{i-1} := G[\{x_1, \ldots, x_{i-1}\}, V(G)]$ to G_i, which is possible as $n \geq h$. ∎

Thus, $e(G) \geq (2n-1) + (h-1)n - \binom{h}{2}$, which implies that $h \leq 3$. Also, $h = 1$ is impossible: colour $G[x_1, \overline{H}]$ red and $G[\overline{H}]$ blue.

Lemma 6 $h = 2$.

Proof. Suppose on the contrary that $h = 3$. Define the partition

$$\Gamma(x_1) \cap \overline{H} = G_\emptyset \cup G_2 \cup G_3 \cup G_{2,3},$$

where, for example, G_2 consists of vertices connected to x_2 but not to x_3 while $G_\emptyset$—to neither x_2 nor x_3. Let $g_\emptyset = |G_\emptyset|$, etc. We want to find non-negative integers $a_\emptyset \leq g_\emptyset$, $a_2 \leq g_2$, $a_3 \leq g_3$, $a_{2,3} \leq g_{2,3}$ such that

$$a_\emptyset + a_2 + a_3 + a_{2,3} \geq d(x_1) - n + 1, \quad (1)$$

$$a_i + a_{2,3} \leq n - 2, \quad i = 2, 3. \quad (2)$$

Case 1 $g_{2,3} \geq n - 1$.
We let $a_\emptyset = g_\emptyset$, $a_2 = g_2$, $a_3 = g_3$, and $a_{2,3} = d(x_1) - n + 1 - g_2 - g_3 - g_\emptyset$. As $d(x_1) \geq g_\emptyset + g_2 + g_3 + g_{2,3}$, we have $a_{2,3} \geq 0$. We have

$$a_2 + a_{2,3} = d(x_1) - g_3 - g_\emptyset - n + 1 \leq (g_2 + g_{2,3} + 2) - n + 1 \leq d(x_2) - n + 3 \leq n - 2.$$

The last inequality above follows from

$$d(x_2) \leq e(G)+3-d(x_1)-d(x_3) \leq e(G)+3-(2n-1)-n \leq 7n/4+O(1). \quad (3)$$

Likewise, $a_3 + a_{2,3} \leq n-2$, as required.

Case 2 $g_{2,3} \leq n-2$.
We let $a_\emptyset = g_\emptyset$, $a_i = \min(g_i, n-3)$, $i = 1,2$, and $a_{2,3} = \min(1, g_{2,3})$. Clearly, (2) is satisfied. We have to check (1). If $g_2, g_3 \geq n-3$, then

$$a_2 + a_3 = 2n - 6 \geq (e(G) - 2n + 3) - n + 1 \geq d(x_1) - n + 1,$$

as required. If, for example, $g_2 \geq n-3 > g_3$, then

$$a_\emptyset + a_2 + a_3 = g_\emptyset + n - 2 + g_3 \geq d(x_1) - g_2 - g_{2,3} + n - 4 \geq d(x_1) - d(x_2) + n - 4,$$

which is at least $d(x_1) - n + 1$ by (3). Finally, (1) holds also if $g_2, g_3 < n-3$:

$$a_\emptyset + a_2 + a_3 = g_\emptyset + g_2 + g_3 \geq d(x_1) - 2 - g_{2,3} \geq d(x_1) - n + 1 - a_{2,3}.$$

Now, that we have found the a's, choose any $A \subset \Gamma(x_1) \cap \overline{H}$ so that $|A \cap G_\emptyset| = a_\emptyset$, $|A \cap G_2| = a_2$, etc. The inequalities (1) and (2) amount to $|A| \geq d(x_1) - n + 1$ and $A \cap \Gamma(x_i) \leq n-2$, $i = 2, 3$.

Colour $G[x_1, A]$ red, while all other edges intersecting $A \cup \{x_1\}$ are blue. At this stage the blue degree of any $x \in H$ is strictly less than n. Now we colour the remaining edges arbitrarily, keeping the red and blue degree of both x_2 and x_3 less than n, which is possible by (3). This colouring contradicts the arrowing property: for example, the red subgraph is a vertex-disjoint union of a star and a graph of maximum degree at most $n-1$. ∎

Lemma 7 $|\Gamma(x_1) \cap \Gamma(x_2)| \geq 2n - 3$.

Proof. If the claim is not true, take a partition $\Gamma(x_1) \cap \Gamma(x_2) = A \cup B$ with $|A|, |B| \leq n-2$. Colour red $G[x_1, \overline{A} \setminus \{x_2\}]$ and $G[x_2, \overline{B} \setminus \{x_1\}]$; all other edges are blue. The blue graph has maximum degree at most $n-1$ while the red graph is a union of two vertex-disjoint stars, a contradiction. ∎

Now, we are able to establish the required lower bounds with little extra effort.

Lemma 8 $\hat{r}(S_{0,n}, S_{1,n}) \geq 2\,|\Gamma(x_1) \cap \Gamma(x_2)| \geq 4n - 6$, *all large* n. ∎

Lemma 9 $\hat{r}(S_{1,n}, S_{2,n}) \geq 4n - 6 + \lceil \frac{2(n-1)}{3} \rceil$, *all large* n.

Proof. Let $A \subset \overline{H}$ be the union of components of $G[\overline{H}]$ which have at least three vertices. Thus $G[B]$ consists of isolated edges and vertices, where $B = \overline{H} \setminus A$. If $|A| \geq n-1$, then the stated bound follows, so assume the contrary.

Colour blue: $G[x_1, A]$, $G[x_2, B]$, and $G[A]$, while all other edges are red. The blue graph does not contain $S_{1,n}$: the component containing x_2 is a star, while any other component has at most $n-1$ vertices. On the other hand, there is no red $S_{2,n}$: $d_{\mathrm{red}}(x_2) \leq n-1$, while there is no red path of length 3 starting at x_1, which is a contradiction. ∎

Lemma 10 $\hat{r}(S_{2,n}, S_{2,n}) \geq 4n - 6 + \lceil \frac{3(n-1)}{4} \rceil$, *all large* n.

Proof. Let $A \subset \overline{H}$ be the union of components of $G[\overline{H}]$ which have at least four vertices. The required lower bound follows unless $|A| \leq n-2$, which we assume.

Colour red: edge x_1x_2 (if it exists), $G[x_1, A]$, $G[x_2, B]$, and a maximal matching in $G[B]$, where $B = \overline{H} \setminus A$. All other edges are blue. The set $\{x_2\} \cup A$, which has at most $n-1$ vertices, is a union of blue components. The remaining blue graph contains no path of length 3 starting at x_1. The red degree of x_1 is at most $|A|+1 \leq n-1$. The red degree of x_2 may be large but there is no red 3-path starting at x_2. Hence, there is no monochromatic $S_{2,n}$, a contradiction. ∎

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