Anti-Ramsey numbers of doubly edge-critical graphs

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Abstract

Given a graph H and a positive integer n, the *Anti-Ramsey number* $\mathrm{AR}(n,H)$ is the maximum number of colors in an edge-coloring of K_n that contains no polychromatic copy of H. The anti-Ramsey numbers were introduced in the 1970's by Erdős, Simonovits, and Sós, who among other things, determined this function for cliques.

In general, few exact values of $\mathrm{AR}(n,H)$ are known. Let us call a graph H *doubly edge-critical* if $\chi(H-e) \geq p+1$ for each edge $e \in E(H)$ and there exist two edges e_1, e_2 of H for which $\chi(H-e_1-e_2) = p$. Here we obtain the exact value of $\mathrm{AR}(n,H)$ for any doubly edge-critical H when $n \geq n_0(H)$ is sufficiently large. A main ingredient of our proof is the stability theorem of Erdős and Simonovits for the Turán problem.

1 Introduction

We will use the standard notation of graph theory, which can be found e.g. in Bollobás' book [3].

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A subgraph of an edge-colored graph is *rainbow* (or *polychromatic*) if all of its edges have different colors. Let $\mathcal{F}$ be a family of graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in $\mathcal{F}$ an *$\mathcal{F}$-free coloring*. The *anti-Ramsey number* $\mathrm{AR}(n, \mathcal{F})$ is the maximum number of colors in an $\mathcal{F}$-free edge-coloring of K_n, the complete graph on n vertices. Anti-Ramsey numbers were introduced by Erdős, Simonovits, and Sós [8]. Various results about this extremal function have been obtained since then: [1, 10, 14, 11, 2, 12, 17, 15, 13, 4] to name a few.

Erdős, Simonovits, and Sós [8] showed that these numbers are very closely related to Turán numbers as follows.

Given a family $\mathcal{F}$ of graphs, let us call a graph G containing no graph in $\mathcal{F}$ as a subgraph an *$\mathcal{F}$-free* graph. Let $\mathrm{EX}(n, \mathcal{F})$ denote the set of $\mathcal{F}$-free graphs on n vertices. (When $\mathcal{F}$ consists of a single graph F, we will write $\mathrm{EX}(n, F)$ instead of $\mathrm{EX}(n, \{F\})$, etc.) The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges of a graph in $\mathrm{EX}(n, \mathcal{F})$. Recall that the *Turán graph* $T_{n,p}$ is a complete p-partite graph on n vertices in which each part has size $\lfloor \frac{n}{p} \rfloor$ or $\lceil \frac{n}{p} \rceil$. Let $t(n, p) = e(T_{n,p})$. Turán [21] showed that $\mathrm{ex}(n, K_{p+1}) = t(n, p)$ and that $T_{n,p}$ is the unique maximum graph in $\mathrm{EX}(n, K_{p+1})$. More generally, the celebrated result of Erdős, Stone, and Simonovits [9, 7] states that for any $\mathcal{F}$ we have

$$\mathrm{ex}(n, \mathcal{F}) = t(n, p) + o(n^2) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2), \tag{1}$$

where $p = \Psi(\mathcal{F})$ is the *subchromatic number*:

$$\Psi(\mathcal{F}) = \min\{\chi(F) : F \in \mathcal{F}\} - 1. \tag{2}$$

Let us give a few other definitions needed to state the results. The *derived family* of $\mathcal{F}$ is

$$\mathcal{F}^- = \{H - e : H \in \mathcal{F},\ e \in E(H)\}.$$

Let xy be an edge in a graph H. Let H_1, H_2 be two copies of $H - xy$. For $i = 1, 2$, let x_i denote the image of x in H_i and y_i the image of y in H_i. Let H^*_{xy} denote the graph obtained by taking H_1 and H_2 and identifying x_1 with x_2 and y_1 with y_2.

Theorem 1 (Erdős, Simonovits, and Sós [8]) *Let $\mathcal{F}$ be an arbitrary graph family. Let $H \in \mathcal{F}$ and $xy \in E(H)$ be such that $\chi(H - xy) = \min\{\chi(F) : F \in \mathcal{F}^-\}$. Then*

$$\mathrm{ex}(n, \mathcal{F}^-) + 1 \le \mathrm{AR}(n, \mathcal{F}) \le \mathrm{ex}(n, H^*_{xy}).$$

It follows from (1) that $\mathrm{AR}(n, \mathcal{F}) = t(n, p) + o(n^2)$*, where* $p = \Psi(\mathcal{F}^-)$.

Proof. We recall the proof because we will need it later.

Let G be a graph in $\mathrm{EX}(n, \mathcal{F}^-)$ of maximum size. Assign distinct colors to all of $E(G)$ and a new color to all of $E(\overline{G})$. This yields an $\mathcal{F}$-free coloring of $E(K_n)$. So $\mathrm{AR}(n, \mathcal{F}) \ge \mathrm{ex}(n, \mathcal{F}^-) + 1$.

Next, suppose that c is an edge-coloring of K_n that uses more than $\mathrm{ex}(n, H^*_{xy})$ colors. Let L be a *representing graph* of c, that is, it is a spanning subgraph of K_n obtained by taking one edge of each color in c (where L may contain isolated vertices). Then L must contain a copy F of H^*_{xy}. This is a rainbow copy of H^*_{xy} in c. Let H_1 and H_2 denote the two copies of $H - xy$ in F as in the definition of H^*_{xy}. Note that H_1 and H_2 are edge-disjoint and since F is rainbow no color appears in both H_1 and H_2. Suppose without loss of generality that color $c(xy)$ does not appear in H_1. Now $H_1 \cup xy$ is a rainbow copy of H. This shows that $\mathrm{AR}(n, \mathcal{F}) \le \mathrm{ex}(n, H^*_{xy})$. ■

Theorem 2 (Erdős, Simonovits, and Sós [8]) *For any $p \ge 2$ and all sufficiently large n, $n \ge n_0(p)$, we have*

$$\mathrm{AR}(n, K_{p+2}) = t(n, p) + 1, \tag{3}$$

and any coloring achieving this bound is obtained by taking a rainbow $T_{n,p}$ and coloring all edges in its complement with the same (extra) color. ■

Montellano-Ballesteros and Neumann-Lara [14] and independently Schiermeyer [17] showed that, in fact, (3) holds for every $n \ge p + 1$ but did not characterize extremal colorings.

Let p be a positive integer. For $0 \le i \le p$ and $n \ge p+i$, let $T^i_{n,p}$ be obtained from the Turán graph by adding a single edge into each of the i largest vertex parts. (Thus $e(T^i_{n,p}) = t(n, p)+i$.)

We say that a graph family $\mathcal{F}$ is *doubly edge-p-critical* if $\Psi(\mathcal{F}^-) \ge p$ but there exist a graph $H \in \mathcal{F}$ and two edges $e_1, e_2 \in E(H)$ such that $\chi(H - e_1 - e_2) = p$. In other words, we require that $T^1_{n,p}$ is $\mathcal{F}$-free for every n but we can add two edges to some $T_{n,p}$ (into two different parts or into the same part) and obtain an element of $\mathcal{F}$ as a subgraph. Furthermore, we subdivide all such families into the following *types*: a doubly edge-p-critical $\mathcal{F}$ has *Type j*, where $j \ge 1$ is the maximum element of $[p]$ such that $T^j_{n,p}$ is $\mathcal{F}$-free for every n. This definition implies that if $\mathcal{F}$ is of Type j, then, for every $n \ge p + j$, we have

$$\mathrm{AR}(n, \mathcal{F}) \ge t(n, p) + j. \tag{4}$$

Indeed, the following coloring of K_n is $\mathcal{F}$-free and uses $t(n, p) + j$ colors.

Construction 3 *Take a copy H of $T_{n,p}$, color its edges with distinct colors and then color its complement $\overline{H}$ with j new colors so that each of the p parts of H is monochromatic and such that each of these j new colors is used on at least one part.*

The main result of our paper shows that this is best possible:

Theorem 4 *Let $p \ge 2$. For an arbitrary (possibly infinite) doubly edge-p-critical family $\mathcal{F}$ of Type j there is an n_0 such that for all $n \ge n_0$ we have*

$$\mathrm{AR}(n, \mathcal{F}) = t(n, p) + j, \tag{5}$$

and all $\mathcal{F}$-free edge-colorings of K_n that achieve this bound are described in Construction 3.

This theorem generalizes Theorem 2 as K_{p+2} (more precisely, $\{K_{p+2}\}$) is doubly edge-p-critical of Type 1 for $p \geq 2$. One can easily generate many examples of doubly edge-critical graphs. One is $T_{n,p}^2$. Further examples can be obtained by taking the join of a doubly edge-critical graph G with an arbitrary graph H. (The *join* operation takes the vertex-disjoint union of G and H and puts the complete bipartite graph in between.) Yet another way is to take a uniquely colorable graph and insert two edges, each inside some color class, etc.

If $\mathcal{F}$ is doubly edge-p-critical, then we can choose a graph $H \in \mathcal{F}^-$ in (3) which is *edge-p-critical* (that is, $\chi(H-e) = p < \chi(H)$ for some $e \in E(H)$). An important result of Simonovits [18] states that $\mathrm{ex}(n, H) = t(n, p)$ for all large n. This was one of our motivations: to prove the analog of Simonovits' result to anti-Ramsey numbers. Our result is a bit more complicated to prove (and to state) but the main idea is the same: we obtain the exact result using the stability approach. The proof can be found in Section 3.

An unpublished conjecture of the first author states that $\mathrm{AR}(n, \mathcal{F}) \leq \mathrm{ex}(n, \mathcal{F}^-) + O(n)$ for any family $\mathcal{F}$ (see [10] for some discussions). Our Theorem 4 confirms the conjecture in the case when, for some $p \geq 2$,

$$\mathrm{ex}(n, \mathcal{F}^-) = t(n, p), \quad \text{for all } n \geq n_0(p, \mathcal{F}). \tag{6}$$

Indeed, any $\mathcal{F}^-$ satisfying (6) has subchromatic number p while some $T_{N,p}^1$ is not $\mathcal{F}^-$-free (for otherwise $\mathrm{ex}(n, \mathcal{F}^-) \geq t(n, p) + 1$ for all n). This shows that $\mathcal{F}$ is doubly edge-p-critical. It is possible that our method might apply in some other cases when $\mathrm{ex}(n, \mathcal{F}^-)$ can be determined via the stability approach. However, it is not clear how to proceed in general.

2 Preliminaries

Here we present some results and definitions that are needed for our proof of Theorem 4 in Section 3.

For a set X and an integer k, let $\binom{X}{k} = \{A \subseteq X : |A| = k\}$. Given a graph G and disjoint subsets $A_1, \ldots, A_p$ of vertices in G, we use $G(A_1, \ldots, A_p)$ to denote the p-partite subgraph of G with parts $A_1, \ldots, A_p$ containing all the edges of G connecting different A_i's. Let $K(A_1, \ldots, A_p)$ denote the complete p-partite graph with parts $A_1, \ldots, A_p$. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S.

Lemma 5 *Let $N, m, p \geq 2$ and $s \geq 0$ be integers. Let G be a p-partite graph with parts $A_1 \cup S, A_2, \ldots, A_p$ where $|S| = s$, $|A_i| = N$ for $i = 1, \ldots, p$, and $S, A_1, \ldots, A_p$ are pairwise disjoint. Suppose that there is no selection of subsets $D_1 \subseteq A_1, D_2 \subseteq A_2, \ldots, D_p \subseteq A_p$ with $|D_i| = m$ for each i such that $G[D_1 \cup S, D_2, \ldots, D_p]$ is a complete p-partite graph. Let $K = K(A_1 \cup S, A_2, \ldots, A_p)$. Let $E^*(S)$ denote the set of edges of K incident to S. Then*

1. $|E(K) \setminus E(G)| \geq \frac{1}{2}(\frac{N}{m})^2$, *or*

2. $|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{1}{2}(\frac{N}{m})$.

Proof. There are $M_1 = \binom{N}{m}^p$ ways to select the D_i's. By our assumption, for each selection, the p-partite graph $G' = G[D_1 \cup S \cup D_2 \cup \ldots \cup D_p]$ misses some edge of K whose two endpoints are in $D_1 \cup S \cup D_2 \cup \ldots \cup D_p$. Suppose first that for at least $\frac{1}{2}M_1$ of these selections, G' misses an edge not incident to S. For each edge $xy \in (E(K) \setminus E(G)) \setminus E^*(S)$, the number ways to select the D_i's such that $x, y \in D_1 \cup \ldots \cup D_p$ is at most $M_2 = \binom{N}{m}^{p-2}\binom{N-1}{m-1}^2$. So, we have

$$|E(K) \setminus E(G)| \geq |(E(K) \setminus E(G)) \setminus E^*(S)| \geq \frac{\frac{1}{2}M_1}{M_2} = \frac{1}{2}\left(\frac{N}{m}\right)^2.$$

For the rest of the proof, suppose that for at least $\frac{1}{2}M$ of the ways to select the D_i's, G' misses an edge of $E(K) \setminus E(G)$ that is incident to S. Fixing an edge $xy \in (E(K) \setminus E(G)) \cap E^*(S)$, where $x \in S$, the number of ways to select the D_i's such that $y \in D_2 \cup \ldots \cup D_p$ is at most $M_3 = \binom{N}{m}^{p-1}\binom{N-1}{m-1}$. This implies that

$$|(E(K) \setminus E(G)) \cap E^*(S)| \geq \frac{\frac{1}{2}M_1}{M_3} = \frac{1}{2}\left(\frac{N}{m}\right).$$

■

Also, we will need the following very useful result.

Theorem 6 (The Stability Theorem, Erdős [5, 6], Simonovits [18]) *Let $\mathcal{F}$ be a family of graphs with subchromatic number p. For every $\epsilon > 0$, there exist $\delta > 0$ and n_ϵ such that for $n > n_\epsilon$ if G is a graph in $\mathrm{EX}(n, \mathcal{F})$ with*

$$e(G) > \mathrm{ex}(n, \mathcal{F}) - \delta n^2,$$

then G can be obtained from $T_{n,p}$ by changing at most ϵn^2 edges. ■

3 Proof of Theorem 4

Let $\mathcal{F}$ be a doubly edge-p-critical family of graphs. We have already demonstrated the lower bound in (5). Let us prove the upper bound and characterize the extremal colorings.

Fix a large integer m, a graph $H \in \mathcal{F}$, and edges $e_1, e_2 \in E(H)$ with $H - e_1 - e_2 \subseteq T_{pm,p}$. Let $\mathcal{F}$ be of Type j. If $j < p$, then (enlarging m if necessary) fix $F \in \mathcal{F}$ such that $F \subseteq T_{pm,p}^{j+1}$. If $j = p$, then we can let e.g. $F = H$.

Given F, H, and m, choose, in this order, small positive constants

$$\frac{1}{m} \gg \epsilon \gg \delta \gg \frac{1}{n_0},$$

where $\alpha \gg \beta$ means that β is sufficiently small depending on α. Let $n \geq n_0$ be arbitrary. We do not specify the dependences between constants explicitly, since we use Theorem 6 as a "black box". In particular, we make no attempt to optimize the constants.

Let c be an extremal $\mathcal{F}$-free edge-coloring of the host graph K_n on the vertex-set V. Let L be a representing graph of c, that is, it is obtained by picking one edge from each color class. By the extremality of c we have

$$e(L) \geq t(n,p) + j. \tag{7}$$

By the proof of Theorem 1, L cannot contain H^*_{xy}. Since $e(L) > t(n,p) > \mathrm{ex}(n, H^*_{xy}) - \delta n^2$, by Theorem 6 L can be obtained from $T_{n,p}$ by changing at most ϵn^2 edges.

Among all complete p-partite graphs with vertex-set V, choose T so that $|E(T) \cap E(L)|$ is maximum. Let the graph T' on V have $E(T) \cap E(L)$ as the edge-set. Our choice of T implies that

$$e(T') \geq t(n,p) - \epsilon n^2. \tag{8}$$

Since $e(T) \leq t(n,p)$, we have

$$|E(T) \setminus E(L)| \leq \epsilon n^2. \tag{9}$$

Let $A_1, A_2, \ldots, A_p$ denote the parts of T. We have, for example, $|A_i| \geq n/2p$ for all $i \in [p]$, for otherwise it is easy to see that the imbalance of the $|A_i|$'s forces $e(T) < t(n,p) - \epsilon n^2$, contradicting (8).

Suppose first that for each $i = 1, \ldots, p$, only one color is used inside A_i.

We apply Lemma 5 with $S = \emptyset$ to T' (or rather to an arbitrary equipartite subgraph of T' with each part having size $N = \lfloor n/2p \rfloor$). By (9), this gives a complete p-partite subgraph $K(D_1, \ldots, D_p) \subseteq T'$ with $|D_i| \geq m + p$ for each $i \in [p]$. Let c_i be the common color of $\binom{D_i}{2}$. Suppose there are exactly r different elements in $\{c_1, \ldots, c_p\}$. By removing p vertices from each D_i, we can find a new complete p-partite graph $K' = T'[D_1 \cup \ldots \cup D_p]$ such that $c(E(K')) \cap \{c_1, \ldots, c_p\} = \emptyset$. It follows that $r \leq j$ for otherwise $j < p$ and G contains a rainbow $T^{j+1}_{mp,p}$, and thus a rainbow F, a contradiction.

Now, (7) implies that $r = j$, $e(T) = t(n,p)$ (and so $T \cong T_{n,p}$), and the coloring c is indeed described by Construction 3.

So, let us assume that some part A_i has at least 2 different colors. We will derive a contradiction. The following observation will be useful in our quest.

Claim 1 There is no complete p-partite graph $K(D_1, \ldots, D_p) \subseteq T'$ such that $|D_i| \geq m + 7$ for each $i \in [p]$ and $c(uv) \neq c(xy)$ for some uv, xy in one part D_i.

Proof of Claim. Suppose that the claim is false. Without loss of generality, $u, v, x, y \in D_1$. We can assume that the pairs uv and xy are disjoint. (Otherwise pick a third pair disjoint from both uv and xy; its color is different from at least one of $c(uv)$ or $c(xy)$.)

By removing 6 vertices from each D_i, we can make sure that no color present in $\binom{\{u,v,x,y\}}{2}$ appears in the new graph $K' = K(D_1, \ldots, D_p)$ (while u, v, x, y are still in D_1). The graph $H \in \mathcal{F}$ can be embedded into $T_{mp,p} + f_1 + f_2$, where f_1, f_2 are two extra edges.

Suppose first that f_1, f_2 are in one part of the Turán graph. If they are disjoint, then we can map f_1, f_2 to uv, xy and extend this mapping to an arbitrary embedding $T_{mp,p} \subseteq K'$. This gives a rainbow copy of H, a contradiction. If $f_1 \cap f_2 \neq \emptyset$, then we can map f_1 to ux and f_2

to one of uv, xy, namely to a pair whose color is different from $c(ux)$. Again this gives us a rainbow H because we took care to eliminate any color in $\binom{\{u,v,x,y\}}{2}$ from $E(K')$.

So let us assume that f_1, f_2 belong to different parts of $T_{mp,p}$. Let us map f_2 to an arbitrary pair wz in D_2. If $c(wz) \in \{c(uv), c(xy)\}$, then map f_1 to one of uv, xy, namely to a pair whose color is different from $c(wz)$. Note that the c-colors of the images of f_1 and f_2 do not appear on $E(K')$, so we can extend this to a rainbow copy of H. If $c(wz) \notin \{c(uv), c(xy)\}$, then at most one edge between wz and $\{u, v, x, y\}$ can have c-color $c(wz)$, and we can safely map f_1 either to uv or xy. At most one edge f of L can have color $c(wz)$. Since we still have some freedom (each current $|D_i| \geq m + 1$) we can find a copy of H avoiding f. This H-subgraph is rainbow, contradicting our assumptions. This finishes the proof of the claim. ■

Call the edges in $E(T) \setminus E(L)$ *missing* and the edges in $E(L) \setminus E(T)$ *bad*. Let M (resp. B) be the graph on V consisting of all missing (resp. bad) edges. Since $e(L) \geq e(T)$, we have

$$e(B) \geq e(M). \tag{10}$$

Let $\gamma = 1/(25pm)$ and $W = \{x : d_M(x) \geq \gamma n\}$.

Claim 2 If uv, xy are any two edges (possibly intersecting) with different colors inside some A_i, then $W \cap \{u, v, x, y\} \neq \emptyset$.

Proof of Claim. For notational convenience, suppose $i = 1$. Let $S = \{u, v, x, y\}$. Let C_1 be a subset of $A_1 \setminus S$ of size $N = \lfloor n/3p \rfloor$; C_1 exists since $|A_1| \geq n/2p$. For $i = 2, \ldots, p$, let C_i be a subset of A_i of size N.

By Claim 1, we cannot find $(m+7)$-sets $D_i \subseteq C_i$, $i \in [p]$, such that $L(D_1 \cup S, D_2, \ldots, D_p)$ is a complete p-partite graph. By (9) and Lemma 5, S is incident to at least $N/(2(m + 7))$ missing edges. Thus at least one vertex of S is incident to at least $N/(8(m + 7)) \geq \gamma n$ edges, as required. ■

Claim 2 in particular implies that $e(M) \geq \gamma n$. Hence, $e(B) \geq \gamma n$. Edges in B have endpoints in the same A_i. Without loss of generality, suppose $B[A_1]$ contains the maximum number of edges of B among all A_i's. Then $e(B[A_1]) \geq e(B)/p \geq \gamma n/p \geq 2$. Let $W_1 = W \cap A_1$. Since $B[A_1]$ has at least two edges (and all edges of $B \subset L$ have different colors), we have $W_1 \neq \emptyset$ by Claim 2. Again, by Claim 2, at most one edge in $B[A_1]$ can be disjoint from W_1. Therefore the edges in $B[A_1]$ can be covered by at most $|W_1| + 1$ vertices in A_1. Let $x \in A_1$ be such that $d_B(x) = \max_{u \in A} d_B(u)$. Then $e(B[A_1]) \leq (|W_1| + 1) \cdot d_B(x)$. On the other hand, there are at least $|W_1| \cdot \gamma n$ edges in M incident to W_1. So, $e(M) \geq |W_1| \cdot \gamma n$. By our assumption, $e(B[A_1]) \geq e(B)/p$. So,

$$|W_1| \cdot \gamma n \leq e(M) \leq e(B) \leq e(B[A_1]) \cdot p \leq (|W_1| + 1) \cdot d_B(x) \cdot p.$$

Let $\alpha = \gamma/2p$. We get

$$d_B(x) \geq \frac{|W_1|}{|W_1| + 1} \cdot \frac{\gamma n}{p} \geq \frac{\gamma n}{2p} = \alpha n.$$

In other words, $|N_L(x) \cap A_1| \geq \alpha n$, where $N_L(x)$ denotes the set of neighbors of x in the graph L. If for some i, $|N_L(x) \cap A_i| < |N_L(x) \cap A_1|$, then by moving x from A_1 to A_i we would strictly increase $e(T')$, contradicting our choice of T. Hence, for each $i = 1, \ldots, p$, $|N_L(x) \cap A_i| \geq \alpha n$. For each $i = 1, \ldots, p$, let C_i be a subset of $A_i \cap N_L(x)$ of size $N = \lfloor \alpha n \rfloor$.

Note that $L(C_1, \ldots, C_p)$ cannot contain a copy of $T_{(m+7)p,p}$. Indeed, since x is connected by an L-edge to every vertex of $C_1 \cup \ldots \cup C_p$, we can swap x with some vertex of $T_{(m+7)p,p}$ in a way that contradicts Claim 1. Applying Lemma 5, with $G = L(C_1, \ldots, C_p)$, $K = T(C_1, \ldots, C_p) \subseteq T$, $N = \lfloor \alpha n \rfloor$, $S = \emptyset$, we conclude that

$$|E(T(C_1, \ldots, C_p)) \setminus E(L(C_1, \ldots, C_p))| \geq \frac{1}{2} \left(\frac{N}{m+7} \right)^2 > \epsilon n^2,$$

which contradicts (9). This completes the proof of Theorem 4. ■

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