Every Tree with at Most 34 Vertices is Prime

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Abstract

A graph of order n is called prime if we can bijectively label its vertices with $\{1, \ldots, n\}$ so that any two adjacent vertices receive coprime labels. Entringer conjectured that any tree is prime. Here we verify this conjecture for all trees with at most 34 vertices. Our proof does not utilize computer search.

1 Introduction

Given a graph G of order n and a set S of n distinct natural numbers, we say that G *admits a prime S-labelling* if there is an S-labelling of G (that is, a bijection $l : V(G) \to S$) such that any two adjacent vertices of G receive coprime labels. The default case is $S = [n] \equiv \{1, \ldots, n\}$ and a graph G admitting a prime $[n]$-labelling is called *prime*.

Around 1980 Entringer conjectured that any tree is prime.

Many classes of trees have been shown to be prime (paths, stars, caterpillars, complete binary trees, spiders); we refer the reader to the dynamic survey by Gallian [2] which contains a section on prime labelling. In particular, Fu and Huang [1] proved that all trees with at most 15 vertices are prime.

Here we verify Entringer's conjecture for all trees with at most 34 vertices. Our proof is not based on computer search and can be easily checked by hand.

The main ingredient of our proof is Lemma 2 which states that for every tree T we can find a vertex u cutting off a subgraph T' whose structure we can control. Namely, we can guarantee that T' either contains exactly 3 vertices or has a very simple structure. Our algorithm labels u with a number coprime to any other in S. Now we can process the components of the forest $T - u$ independently. We label T' (typically by some of the larger elements of S)

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knowing its structure; we label $T - T' - u$ with the remaining labels using induction.

Unfortunately, we cannot always guarantee that the remaining labels form an interval of integers; this is why we introduce the following predicate $P(S)$: *"Every tree with $|S|$ vertices admits a prime S-labelling."*

Thus, here we establish the validity of $P_i \equiv P([i])$ for all $i \leq 34$. Let $I(i)$ be the induction step: *"If every tree with fewer than i vertices is prime, then every tree of order i is prime."* In our notation,

$$I(i) = P_i \vee (\neg P_1) \vee \cdots \vee (\neg P_{i-1}).$$

2 Warm-Up

A greatly simplified version of the above principle can be recognized in the proof of the following lemma.

Lemma 1 *Let $p \geq 3$ be a prime. Then $I(p)$, $I(p+1)$ and $I(p+2)$ are valid.*

Proof. Suppose that every tree with at most $p-1$ vertices is prime.

Every tree of order p is prime: assign p to some endvertex and label the remaining graph by the assumption. Similarly, labelling an endvertex by $p+1$ and its neighbour by p, one proves P_{p+1}.

Finally, let T be a tree with $p+2$ vertices. Let $(x_1, \ldots, x_m)$ be a longest path in T. If $d(x_2) = 2$, then we let $l(x_1) = p+2$, $l(x_2) = p+1$ and $l(x_3) = p$. Otherwise, let $l(x_1) = p+2$, $l(x_2) = p$ and $l(v) = p+1$, where v is a neighbour of x_2 distinct from x_1 and x_3; by the choice of the path, v is an endvertex. In both cases our partial labelling extends to a prime labelling of T. ∎

3 Key Lemma

Here is the promised result.

Lemma 2 *Let T be a tree with at least 4 vertices. Then there is a vertex u such that either some set A of 3 vertices can be represented as a union of (one or more) components of $T - u$ or there are a neighbour v of u of degree $k+1 \geq 3$ such that its other k neighbours $v_1, \ldots, v_k$ have degree 2 each and are incident to endvertices $u_1, \ldots, u_k$ correspondingly.*

Proof. Let $(x_1, \ldots, x_m)$ be a longest path in T.

If $d(x_2) \geq 4$ or if $d(x_3) = 1$, then x_2 is connected to at least 3 endvertices and we are done (take $u = x_2$). Suppose that $d(x_2) \leq 3$ and $d(x_3) \geq 2$.

If $d(x_2) = 3$, then x_2 is incident to precisely 2 endvertices and we can take $u = x_3$ here. Let $d(x_2) = 2$.

Let b be any neighbour of x_3 different from x_4. If $d(b) = 1$, then $u = x_3$ with $A = \{x_1, x_2, u\}$ does the job. By the choice of the path, all neighbours of b

are endvertices. If $d(b) = 3$, we can take $u = x_3$. If $d(b) \geq 4$, we can take $u = b$. Thus, we may assume that $d(b) = 2$.

Finally, if $d(x_3) = 2$, let $u = x_4$ (and $A = \{x_1, x_2, x_3\}$); otherwise we find the second configuration with $u = x_4$ and $v = x_3$. ∎

4 Labelling Table and its Properties

To prove that every tree with at most 34 vertices is prime, we validate each $I(i)$, $i \leq 34$.

Lemma 1 takes care of most of the cases except for $i \in \{10, 16, 22, 26, 27, 28, 34\}$. In each of these cases, we will follow the plan outlined in the introduction. Unfortunately, there is not a single partial labelling which works for each i. So we wrote up a little table which contains directions how to label configurations obtained by Lemma 2 in each case.

This will be described later; at the moment we claim that Table 1 enjoys all of the following properties.

1. Each $S = [i]$ for $i \in \{10, 16, 22, 26, 27, 28, 34\}$ is present.
2. In each triple $l(A) \subset S$, there is a number coprime to the other two.
3. $l(u) \in S$ is coprime to any other element of S.
4. $l(v) \in S$ is coprime to each $l(u_j)$, $1 \leq j \leq k$.
5. For each $j \in [k]$, $l(u_j) \in S$ and $l(v_j) \in S$ are coprime.
6. $S \setminus (A \cup \{l(u)\})$ either equals some $[m]$ or can be found in Table 1.
7. $S \setminus (\{l(u), l(v)\} \cup (\cup_{j=1}^{k} \{l(u_j), l(v_j)\}))$ equals some $[m]$.

This looks like a long list but each property can be easily checked by hand.

5 Putting all together

Now we are ready to prove our main result which we state as follows.

Theorem 3 *The property $P(S)$ holds for every $S = [i]$ with $1 \leq i \leq 34$ and for every S appearing in Table 1.*

Proof. We use induction on $|S|$. Trivially P_1 and P_2 hold.

If $S = [i]$ with $0 \leq i - p \leq 2$ for some prime p, then we are home by Lemma 1. Otherwise S appears in Table 1.

Let T be any tree of order $|S| \geq 4$. Apply Lemma 2 to T.

If the first alternative of Lemma 2 holds, then in the corresponding row we find the label for u (column $l(u)$) and three labels (column $l(A)$) to be used for A. Of these three labels, one is coprime to the other two and, as there are at most two edges of T inside of A, we can find a prime labelling of $T[A]$ with

S	$l(A)$	$l(u)$	k	$l(v)$	$(l(u_j))_{1\leq j\leq k}$	$(l(v_j))_{1\leq j\leq k}$
$[10]$	8, 9, 10	7	≥ 2	8	$9,\ (9-2j)_{2\leq j\leq k}$	$10,\ (10-2j)_{2\leq j\leq k}$
$[16]$	14, 15, 16	13	≥ 2	16	$15,\ 11,\ (15-2j)_{3\leq j\leq k}$	$(16-2j)_{1\leq j\leq k}$
$[22]$	20, 21, 22	19	2	22	21, 17	20, 18
			≥ 3	16	$21,\ 15,\ 17,\ (21-2j)_{4\leq j\leq k}$	$20,\ 22,\ 18,\ (22-2j)_{4\leq j\leq k}$
$[26]$	24, 25, 26	23	≤ 4	26	$25,\ (25-2j)_{2\leq j\leq k}$	$(26-2j)_{1\leq j\leq k}$
			≥ 5	16	$25,\ (25-2j)_{2\leq j\leq k}$	$24,\ 26,\ 20,\ 18,\ 22,\ (26-2j)_{6\leq j\leq k}$
$[17]\cup\{26,27\}$	16, 26, 27	17	≥ 2	16	$27,\ (19-2j)_{2\leq j\leq k}$	$26,\ (18-2j)_{2\leq j\leq k}$
$[17]\cup\{19,20,21,22,26,27\}$	20, 21, 22	19	2	26	27, 21	22, 20
			≥ 3	16	$27,\ 21,\ 17,\ (23-2j)_{4\leq j\leq k}$	$26,\ 22,\ 20,\ (22-2j)_{4\leq j\leq k}$
$[27]$	18, 24, 25	23	≥ 2	23	$27,\ 25,\ (27-2j)_{3\leq j\leq k}$	$(28-2j)_{1\leq j\leq k}$
$[22]\cup\{24,25\}$	18, 24, 25	19	≤ 3	22	$25,\ 21,\ (17)_{j=3}$	$24,\ 20,\ (18)_{j=3}$
			≥ 4	16	$25,\ 21,\ 17,\ 15,\ (23-2j)_{5\leq j\leq k}$	$24,\ 20,\ 18,\ 22,\ (24-2j)_{5\leq j\leq k}$
$[28]$	26, 27, 28	23	≤ 6	26	$27,\ 25,\ (27-2j)_{3\leq j\leq k}$	$28,\ (28-2j)_{2\leq j\leq k}$
			≥ 7	16	$27,\ 21,\ 25,\ 15,\ 17,\ 19,\ (27-2j)_{7\leq j\leq k}$	$(30-2j)_{1\leq j\leq 6},\ (28-2j)_{7\leq j\leq k}$
$[17]\cup\{20,21,22\}$	20, 21, 22	17	≥ 2	16	$21,\ 15,\ (19-2j)_{3\leq j\leq k}$	$20,\ 22,\ (20-2j)_{3\leq j\leq k}$
$[34]$	32, 33, 34	31	≥ 2	32	$33,\ (33-2j)_{2\leq j\leq k}$	$34,\ (34-2j)_{2\leq j\leq k}$

Table 1: Labelling Table.

these labels. There are no edges between A and $V(T) \setminus (A \cup \{u\})$ and $l(u)$ is coprime to any other element of S. Thus, the task reduces to labelling the forest $T - A - u$ with $S \setminus (l(A) \cup \{l(u)\})$, which can be done by induction because $S \setminus (l(A) \cup \{l(u)\})$ is either an interval or can be found in Table 1.

The second alternative of Lemma 2 is treated in the same way except we may have a few different label assignments depending on k. Also, if we have $l(u) = l(v)$ in the table, then we assign this label to v leaving u unlabelled. ∎

Remark. Of course, the validity of $I(10)$ follows from the result of Fu and Huang [1], but we get it at almost no extra cost while keeping our paper self-contained.

Remark. Our proof gives a practical algorithm for labelling trees with at most 34 vertices.

6 Some Final Remarks

In the next open case $P([35])$ we were not able to find a proof within the above framework, so we do not know if our method works here (our search was not extensive).

As we see, in attacking Entringer's conjecture it was helpful to introduce the more general property $P(S)$. Unfortunately, the natural generalization that $P(\{i, i+1, i+2, \ldots, j\})$ holds for any $i < j$ is not correct as the following example demonstrates.

Let $K_{1,18}$ be the star with 18 edges, that is, some vertex, called the *centre*, is connected to any other of 18 remaining vertices. One can choose i such that i equals 0 (mod 2), 0 (mod 3), 4 (mod 5), 1 (mod 7), 4 (mod 11), 8 (mod 13) and 0 (mod 17). Then the consecutive members of $S = \{i, i+1, \ldots, i+18\}$ are respectively divisible e.g. by

$$17, 5, 2, 3, 2, 13, 7, 11, 2, 3, 2, 5, 2, 7, 2, 3, 2, 17, 11 \cdot 13,$$

and there is no suitable label for the centre.

This indicates that it is probably hard to characterize all such S for which $P(S)$ holds. Salmasian [3] investigated the property $P'(n, S)$: "*Any tree T of order n admits a prime S'-labelling for some $S' = S'(T) \subset S$*" and proved that $P'(n, [4n])$ holds for all $n \geq 50$. Of course, this all makes Entringer's conjecture even more interesting.

References

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