Reading module on quantum field theory

Term 2, 2020-2021

Moderator: Minhyong Kim

Textbook: Folland, Gerald B. Quantum field theory. A tourist guide for mathematicians. Mathematical Surveys and Monographs, 149. American Mathematical Society, Providence, RI, 2008. xii+325 pp.

Fields in physics refer to various quantities varying continuously over spacetime, such as a scalar field (just a function), the electromagnetic field (a vector field or a two-form), or the wave function of a relativistic electron (spinor field). Classical field theory studies the space of solutions to the differential equations satisfied by such field, such as the Klein-Gordon equation, the Maxwell equation, or the Dirac equation. Quantum field theory is literally the quantum theory of such fields, whereby the (components of the) field values become operators acting on a Hilbert space $\mathcal{H}$ of possible states of the field. Given a space of fields ϕ, the main goal of quantum field theory is to compute the *vacuum expectation values of products of field values*

$$\langle \Omega, \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)\Omega\rangle.$$

Here, $\hat{\phi}(x)$ is the operator that quantizes the possible values of the field ϕ at the spacetime point x and $\Omega \in \mathcal{H}$ is the so-called 'vacuum state'. Such a theory is necessary for understanding the detailed structure of matter and their interactions such as the emission and absorption of light by atoms or the creation and annihilation of particles.

The goal of this course is to come to a basic understanding of these foundations as well as the practical toolkit of the calculating physicist following the book above, which presents the material in a manner accessible to people with an advanced undergraduate-level mathematical background and general knowledge of physics. It is expected that the student will work through the first six chapters of the book in the course of the term.

Prerequisites

–Classical mechanics including the Hamiltonian and Lagrangian formalism;
–Electromagnetism including some familiarity with special relativity;
–Quantum mechanics, especially a thorough understanding of the harmonic oscillator;
–Fourier analysis, i.e, Fourier series, Fourier transforms, and some bits of functional analysis surrounding the theory;
–Lie theory: Lie groups and Lie algebras, especially the group $SL_2(\mathbb{C})$ and its Lie algebra.

Parts of the prerequisites can be acquired during the term provided the student is willing to read very actively and independently. It is suggested that the book be examined before registering for the course to gauge the feasibility of this approach.

Assessment

There will be a 3-hour examination in the examination session following Easter vacation.

Review of textbook from MathSciNet

Quantum field theory is a powerful theoretical framework in high-energy physics that describes all known elementary particles and their interactions as they have been observed in collision experiments and cosmological observations. Its origins can be traced back to the early days of quantum mechanics, which was set on a thoroughly mathematical footing mainly by von Neumann in the early 1930s. However, many of the techniques used in quantum field theory still lack a mathematical formulation and remain formal, in remarkable contrast with the precision of its physical predictions. This motivates many mathematical physicists to explore or search for the underlying mathematical theory. The present "tourist guide to quantum field theory" is a great contribution to explaining to a more general audience of mathematicians what this search is all about. It lets the reader approach physics as tourists would "approach a foreign country, as a place to enjoy and learn from but not to settle in permanently" [p. x]. Indeed, the physical ideas presented in this monograph might serve as a source of inspiration for mathematicians. As we know, already many ideas from quantum field theory have led to huge leaps forward in pure mathematics. For instance, Donaldson's classification of 4-manifolds heavily relies on instantons, which are special field-theoretical configurations. Also, knot theory has received great stimulus from Chern-Simons theory in constructing natural invariants of 3-manifolds, and many other examples exist. The style of the present monograph is clear and the author is honest about possible mathematical shortcomings of quantum field theory; after all, it is intended for a mathematical audience. The material in the book is the typical content for five years of theoretical (particle) physics studies, ranging from classical mechanics, via Einstein's special theory of relativity, to quantum mechanics and eventually to quantum gauge fields and the Standard Model of high-energy physics. This makes the book a pleasure to read for people with some background in physics, as the prerequisites demand. But even those without such a background will find the book of great value: throughout there are many connections to the physics literature, especially for mathematicians with little or even no background in physics. After a first chapter containing a dictionary with physics and mathematics terminology of the typically identical concepts, a summary is presented of classical mechanics and basic quantum mechanics. Clearly, since the mathematical theory is available, this is done in a rigorous manner, exploiting powerful techniques and results from functional analysis. This continues in Chapter 4, where relativistic quantum mechanics is described in great detail, centered around the Dirac equation. Chapter 5 deals with free quantum fields, for which a mathematical theory actually exists. This involves operator-valued distributions and appears in Section 5.2, after a more formal discussion of the same theory in Section 5.1 (thereby preparing for the rest of the book). The chapter ends with a summary of the Wightman axioms as well as the related approach taken in algebraic quantum field theory. Both are axiomatic schemes designed to describe quantum fields. Unfortunately, in dimension 4 the only known examples of these schemes are free field theories; in lower dimensions interacting theories have been obtained as well. Naturally, the rest of the book is of a different nature. Chapter 6 forms the core of the book with its description of quantum fields with interactions, and as the author puts it: "to make further progress, however, it is necessary to make a bargain with the devil" [p. 123]. We thus leave the solid mathematical ground and exchange it for the, in this case, more powerful formal methods of perturbation theory. The lack of convergence of the perturbation series is what is so devilish about it, and it signifies a turning point in the style of the book. Nevertheless, I would still suggest this as the place for a mathematician to learn quantum field theory, mainly because it stays close to the mathematical setup of the previous chapters. At this point, the more conventional quantum field theory textbooks quickly embrace the ill-defined path integral, thereby cloaking the mathematical structure. However, since these functional integrals are in fact the more powerful methods for modern quantum (gauge) field theory, a concise overview is given in Chapter 8. Chapter 7 gives an exposition of the process of renormalization. Unfortunately, this appears to most mathematicians as an obscure process, consisting of an effort to make sense of divergent integrals. Rather, it expresses that at present we do not have a fully consistent mathematical theory describing quantum field theory. In a search for such a theory, it is therefore inevitable to study the process of renormalization. Actually, in itself, renormalization is of combinatorial nature and can thus be made quite rigorous. This chapter ends with the derivation of probably the most impressive predictions of quantum electrodynamics: the anomalous electromagnetic moment of the electron (up to one-loop corrections). The last chapter forms an invitation to the quantum field theories that constitute the so-called Standard Model of high-energy physics, namely, quantum gauge theories. The Standard Model describes all the known particles and their interactions. Again, the author shows his ability to "speak both languages", presenting in parallel the necessary concepts in both the physical and mathematical vocabulary. This chapter is short for such a broad topic; however, the many references allow the reader to find a path into the literature. Reviewed by Walter D. van Suijlekom

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