SHIMURA VARIETIES. EXERCISES 7.

(1) Show that every pure $\mathbb{Q}$-Hodge structure of dimension two is polarisable.

Show that there exist non-polarisable four dimensional $\mathbb{Q}$-Hodge structures (find one of type $(-1,0)+(0,-1)$).

(2) Let V be a $\mathbb{Q}$-Hodge structure. Show that

$$MT(V^n) = MT(V)$$

(Here $MT(V)$ is embedded in $GL(V)$ via the diagonal).

(3) Let $V_1, \dots, V_r$ be $\mathbb{Q}$-Hodge structures. Show that

$$MT(V_1 \times \cdots \times V_r) \subset MT(V_1) \times \cdots \times MT(V_r)$$

and the projections $MT(V_1 \times \cdots \times V_r) \longrightarrow MT(V_i)$ is surjective.

(4) Let $n \in \mathbb{Z}$, $n \neq 0$.

Show that $MT(\mathbb{Q}(n)) = \mathbb{G}_m$.

(5) Let F be a number field.

Show that $\prod_v F_v$ is not locally compact while $\prod'_v F_v$ is.

(6) Let $F = \mathbb{Q}$.

Recall that $\mathbb{A}_\infty = \mathbb{R} \times \prod_p \mathbb{Q}_p$.

Show that

$$\mathbb{A} = \mathbb{Q} + \mathbb{A}_\infty$$

(Hint: Chinese remainder theorem.)

Generalise to a general number field F.

(7) Show that the restricted product topology on $\mathbb{A}_F^*$ is not the same as the subspace product topology induced by $\mathbb{A}_F^* \subset \mathbb{A}$.

(find a sequence converging in one topology but not in the other)

(8) Let F be a number field. Show that F^* embedded diagonally in $\mathbb{A}_F^*$ is a discrete subgroup.

Show that for any valuation v, F_v^* is a closed subgroup.

(9) Let F be a number field and let $Cl(F)$ be the class group of F. Define the map

$$\mathbb{A}_F^* \longrightarrow Cl(F)$$

$$(a_v) \mapsto I = \{x \in F : |x|_v \leq |a_v|_v \text{ for all finite} v\}$$

Show that the map is well defined and surjective.

Show that it induces an isomorphism

$$\mathbb{A}_F^*/F^*(F \otimes \mathbb{C})^* O_F^* \cong Cl(F)$$