**Shimura Varieties: Problem sheet 6**

# Tori and Hodge structures

12 November 2014

1. **Representations of tori**

   Let $K$ be a field of characteristic zero.

   (a) Let $T$ be a torus over the algebraic closure $\bar{K}$. Show that torsion points are Zariski dense in $T(\bar{K})$.

   (b) Let $\rho : T \to \mathrm{GL}(V)$ be a finite-dimensional representation of $T$ (we require that $\rho$ is a morphism of algebraic groups). Show that we can choose a basis for $V$ with respect to which, for all torsion points $x \in T(\bar{K})$, $\rho(x)$ is diagonal.

   (c) Deduce from (i) and (ii) that, with respect to a suitably chosen basis, the image $\rho(T)$ is contained in the diagonal matrices in $\mathrm{GL}(V)$.

   (d) Deduce that $\rho$ is isomorphic to a direct sum of one-dimensional representations.

   In other words, each isomorphism class of representations of $\rho$ is isomorphic to
   $$\bigoplus_{\chi \in X^*(T)} \chi^{m(\rho,\chi)}$$
   for some function $m(\rho, -)\colon X^*(T) \to \mathbb{Z}_{\geq 0}$ which is zero for all but finitely many $\chi$.

   (e) Now let $T$ be a torus over $K$ itself. Let $\rho\colon T \to \mathrm{GL}(V)$ be a representation of $T$ which is defined over $K$ (as a morphism of algebraic groups. Prove that
   $$m(\rho, \sigma\chi) = m(\rho, \chi)$$
   for all $\chi \in X^*(T)$ and $\sigma \in \mathrm{Gal}(\bar{K}/K)$.

   (f) Prove that there is a natural bijection between irreducible representations of $T$ defined over $K$ and $\mathrm{Gal}(\bar{K}/K)$-orbits in $X^*(T)$.

2. **Representations of the Deligne torus**

   (a) Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ be the Deligne torus. Label the standard characters of $\mathbb{S}$ which generate its character group as $\chi$ and $\bar{\chi}$.

   Describe the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits in $X^*(\mathbb{S})$, and the associated $\mathbb{R}$-irreducible representations of $\mathbb{S}$.

We define an $\mathbb{R}$-**Hodge structure** to be a finite-dimensional real vector space $V_\mathbb{R}$ together with a direct sum decomposition

$$V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q}$$

satisfying

$$V^{q,p} = \overline{V^{p,q}}$$

for all $p, q$ (where the bar denotes complex conjugation).

(b) Define a representation $\mathbb{S}(\mathbb{C}) \to \mathrm{GL}(V_\mathbb{C})$ by letting $\mathbb{S}(\mathbb{C})$ act on $V^{p,q}$ via the character $\chi^{-p}\bar{\chi}^{-q}$. Prove that this defines a real representation $\mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$. (The minus signs in the exponents here are unimportant, but they are a standard convention.)

(c) Show that there is an equivalence of categories between $\mathbb{R}$-Hodge structures and representations of $\mathbb{S}$ defined over $\mathbb{R}$.

3. **Tori over $\mathbb{Q}$**

Let $F$ be a number field. We define a $\mathbb{Q}$-torus $T_F$ (also known as $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{m,L}$) as follows:

Let $\Psi$ be the set of embeddings $F \to \bar{\mathbb{Q}}$. There is a natural action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\Psi$.

Let $\mathbb{Z}[\Sigma]$ be the free $\mathbb{Z}$-module of formal linear combinations of elements of $\Sigma$:

$$\mathbb{Z}[\Sigma] = \{a_1\sigma_1 + \cdots + a_n\sigma_n \mid (a_i) \in \mathbb{Z}^n\}$$

where $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$. Define an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{Z}[\Sigma]$ by linearly extending its action on $\Sigma$.

Then $T_F$ is the $\mathbb{Q}$-torus whose character group is $\mathbb{Z}[\Sigma]$ with the given Galois action.

(a) Prove that the group of $\mathbb{Q}$-points of $T_F$ is naturally isomorphic to $F^\times$., and more generally, $T_F(A) = (F \otimes_\mathbb{Q} A)^\times$ for any $\mathbb{Q}$-algebra $A$.

(b) Let $n = [F : \mathbb{Q}]$ and choose a basis $\{e_1, \ldots, e_n\}$ for $F$ as a $\mathbb{Q}$-vector space. Consider the homomorphism $F^\times \to \mathrm{GL}_n(\mathbb{Q})$ which sends $x \in F^\times$ to the linear map "multiply by $x$ in $F$", with respect to the chosen basis. Prove that this is the map on $\mathbb{Q}$-points induced by a morphism of $\mathbb{Q}$-algebraic groups $T_F \to \mathrm{GL}_{n,\mathbb{Q}}$.

This realises $T_F$ as a $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}_n$. For example, choosing the basis $\{1, \sqrt{d}\}$ for the quadratic field $\mathbb{Q}(\sqrt{d})$, we get

$$T_F \cong \left\{ \begin{pmatrix} x & dy \\ y & x \end{pmatrix} \in \mathrm{GL}_2 \right\}.$$

(c) Prove that the splitting field of $T_F$ is the Galois closure of $F$.

(d) Consider the character $\sigma_1 + \cdots + \sigma_n$ (where $\Sigma = \{\sigma_1, \ldots, \sigma\}$). Observe that this character is invariant under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and deduce that it defines a morphism of $\mathbb{Q}$-algebraic groups

$$N\colon T_F \to \mathbb{G}_{m,\mathbb{Q}}.$$

Show that its value on $\mathbb{Q}$-points is the same as the norm $\mathrm{Nm}_{F/\mathbb{Q}}$.