Shimura Varieties: Problem sheet 5

Reductive groups

November 3, 2014

Throughout, we let K denote an arbitrary field of characteristic 0. Algebraic groups are considered over K unless specified otherwise.

<u>Exercise I:</u> Tori.

An algebraic group T over K is a **torus** if there exists an integer n and a Galois extension L/K such that $T \times_K L \cong \mathbb{G}_{\mathrm{m},L}^n$.

Recall the character group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ and cocharacter group $X_*(T) = \mathrm{Hom}(\mathbb{G}_\mathrm{m}, T)$. There is a pairing $\langle -, - \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ given by composition: $\alpha \circ \mu = \left(x \mapsto x^{\langle \alpha, \mu \rangle}\right) \in \mathrm{Hom}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m})$ for $\alpha \in X^*(T)$, $\mu \in X_*(T)$.

(a) Let T be a torus, with an isomorphism $i \colon T_0 = \mathbb{G}_{\mathrm{m},L}^n \to T \times_K L$ for some finite Galois extension L/K.
Prove that for every $\sigma \in \mathrm{Gal}(L/K)$, conjugation by σ gives rise to an automorphism a_σ of T_0 through the formula $\sigma(i) = i \circ a_\sigma$.

(b) Show that the assignment $\sigma \mapsto a_\sigma$ is a **1-cocyle**: $a_{\sigma\tau} = a_\sigma \sigma(a_\tau)$.

(c) Use this to construct a homomorphism $\rho_T \colon \mathrm{Gal}(L/K) \to \mathrm{GL}_n(\mathbb{Z})$. (Hint: exercise 8 from the previous sheet tells us we can identify automorphisms of T_0 with automorphisms of $X^*(T_0)$, both over L and over K.)

(d) Show that choosing a different isomorphism $j = i \circ b$ instead of i, giving a 1-cocycle a', changes a by the following formula: $a'_\sigma = b^{-1} a_\sigma \sigma(b)$. Deduce that ρ_T is uniquely determined up to conjugacy.

(e) Conversely, given a Galois representation $\rho \colon \mathrm{Gal}(L/K) \to \mathrm{GL}(M)$, show that $T(K) = (L^\times \otimes_\mathbb{Z} M^\vee)^{\mathrm{Gal}(L/K)}$ defines a torus T over K with $T \times_K L \cong \mathbb{G}_\mathrm{m}^n$ and $\rho_T = \rho$ (up to conjugacy). (Here $M^\vee$ is the dual representation to ρ, and $\mathrm{Gal}(L/K)$ acts diagonally on the tensor product $L^\times \otimes_\mathbb{Z} M^\vee$.)

(f) Apply this result to classify all tori over the field $K = \mathbb{R}$. (Hint: $\mathbb{Z}/2\mathbb{Z}$ admits exactly three isomorphism classes of indecomposable integral representations. Which are they?)

<u>Exercise II:</u> Rank zero semisimple groups.

An algebraic group is called **semisimple** if it has no nontrivial solvable normal subgroups. The **rank** of a semisimple group G is the maximal dimension of a torus inside G.

Show that a connected semisimple group of rank 0 is trivial.

<u>Exercise III:</u> Roots and coroots.

Let D be the subgroup of diagonal matrices of SL_2.

(a) Show that $X^*(D)$ is generated by $\lambda\colon \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \mapsto x$, and $X_*(D)$ is generated by $\mu\colon x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$.

(b) Let D act on the Lie algebra $\mathfrak{sl}_2$ of SL_2 (consisting of traceless 2×2 matrices), by conjugation. Which characters of $X^*(D)$ appear in the decomposition of this representation of D into irreducible representations? Such characters which are nonzero are called **roots**.

(c) Now consider $G = \mathrm{PGL}_2$ and take T to consist of the diagonal subgroup. Write down $X^*(T)$, $X_*(T)$ and the subset of roots $\Phi \subset X^*(T)$, obtained by decomposing the conjugation action of T on $\mathfrak{pgl}_2 = \mathfrak{gl}_2/K$.

(d) For each root α, write down a nonzero homomorphism $u_\alpha\colon \mathbb{G}_\mathrm{a} \to G$ such that $tu_\alpha(a)t^{-1} = u_\alpha(\alpha(t)a)$ for all $t \in T$, $a \in \mathbb{G}_a$. Prove that the image of such a homomorphism is necessarily uniquely determined by this condition: this is the **root group** U_α.

(e) Prove that for each root $\alpha \in \Phi$, there exists a homomorphism $\varphi_\alpha\colon \mathrm{SL}_2 \to G$ that sends D into T, and maps $U^+ \subset \mathrm{SL}_2$ (respectively U^-) isomorphically onto U_α (respectively, $U_{-\alpha}$). Here $U^+ = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right\} \subset \mathrm{SL}_2$, and similarly $U^- = \left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right\}$.

(f) Define the **coroot** $\alpha^\vee$ attached to the root α as the cocharacter given by $\alpha^\vee(x) = \varphi_\alpha\left(\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}\right)$, i.e. $\alpha^\vee = \varphi_\alpha \circ \mu$. Show that this is well defined (i.e. independent of any choice of φ_α), and compute $\langle \alpha, \beta^\vee \rangle$ for $\alpha, \beta \in \Phi$.

(g) Bonus exercise! Carry out these calculations for the following algebraic groups: SL_3, Sp_4, $\mathrm{SL}_2 \times \mathrm{SL}_2$ and SO_5. Compare results with your answer to the last part of exercise IV.

Exercise IV: Root systems.

We can construct roots and coroots for more general groups than we did in the previous exercise. The algebraic structure of the resulting object is that of a **root datum**. This consists of:

- Free abelian groups of finite rank M and $M^\vee$, with finite subsets of nonzero elements $\Phi \subset M$ (the roots) and $\Phi^\vee \subset M^\vee$ (the coroots).
- A perfect pairing $\langle -, - \rangle \colon M \times M^\vee \to \mathbb{Z}$.
- A bijection $\alpha \leftrightarrow \alpha^\vee$ between Φ and $\Phi^\vee$.

This data is subject to the following conditions:

- $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$,
- $s_\alpha(\Phi) = \Phi$, where $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ is the "reflection in the hyperplane perpendicular to $\alpha^\vee$", and dually for $s_{\alpha^\vee}$.

Moreover, a root datum is said to be **reduced** if, for every $\alpha \in \Phi$, $n\alpha \in \Phi \iff n \in \{\pm 1\}$. Given a (reduced) root datum, one can form the associated (reduced) **root system** by considering the real vector space $V = M \otimes \mathbb{R}$ with $\Phi \subset V$.

Assume from now on that the roots span V (this is the semisimple case).

(a) Define a pairing $(-, -)$ on V by the formula $(\alpha, \beta) = \sum_{\lambda \in \Phi} \langle \alpha, \lambda^\vee \rangle \langle \beta, \lambda^\vee \rangle$. Prove that this pairing makes V into an Euclidean vector space, i.e. that the pairing is symmetric and positive definite.

(b) Prove that $\langle \alpha, \beta^\vee \rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$. Deduce that the orthogonal projection of any root α onto another root β is always a half-integral multiple of β, and also that the reflections s_α are orthogonal with respect to $(-, -)$.

(c) Let θ be the angle between any two roots, as defined using $(-, -)$.
Show that $\theta \in \{0, \pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}\} + \pi\mathbb{Z}$.

(d) Let $\Delta \subset \Phi$ be a **simple subsystem**: a linearly independent subset such that every element α of Φ can be (uniquely) written as a linear combination $\alpha = \sum_i a_i \delta_i$ with $\delta_i \in \Delta$, with the condition that either $a_i \geqslant 0$ for all i, or that $a_i \leqslant 0$ for all i.
Show that the angles between any two vectors in a simple subsystem are obtuse.

(e) Let Φ^+ be any subset of Φ with $\Phi = \Phi^+ \coprod -\Phi^+$. This is a choice of **positive** roots. Prove that there exists a unique simple subsystem $\Delta \subseteq \Phi^+$.

(f) Classify all rank 2 (reduced) root systems. (Hint: It suffices to consider a simple subsystem. Which pairs of vectors in the plane can form a root system?)