Shimura Varieties: Lecture 1

Modular curves

8 October 2014

Martin Orr

Introduction to course
Motivation for modular curves, mention some applications
- Modular forms
- Moduli of elliptic curves, Mazur's theorem on torsion
- Modularity theorem

Modular curves as Riemann surfaces
- The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$
- Fundamental domain, elliptic points
- Congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, quotients $\Gamma\backslash\mathcal{H}$
- Cusps, compactification as a Riemann surface
- Topological proof that $X(1) \cong \mathbb{P}^1$, define j by its values at elliptic points

Elliptic curves as $\mathbb{C}/\Lambda$ and Weierstrass equation
Interpretation of $Y(1)$ as coarse moduli space of elliptic curves
- Getting τ from a lattice
- Families of elliptic curves (analytically)
- Period map

Shimura Varieties: Lecture 2

Modular curves as moduli spaces, Ideals in Dedekind domains

15 October 2014

Martin Orr

Ring of integers of a number field
Prime factorisation of ideals
Dedekind domains
Ring of regular functions on an affine algebraic curve (over $\mathbb{C}$)
Divisors/fractional ideals
Class group/Picard group

Moduli interpretation for $Y_1(N)$
    Torsion on elliptic curves
Moduli interpretation for $Y(N)$
    Weil pairing (mentioned briefly)

Shimura Varieties: Lecture 3

Models of modular curves over $\mathbb{Q}$, Local fields

22 October 2014

Martin Orr

Discrete valuations: order of a function at a point and $\mathfrak{p}$-adic valuations
Completions
- Formal power series over $\mathbb{C}$
- $\mathfrak{p}$-adic completions of number fields

The equivalence classes of absolute values on a function field and on a number field

Moduli spaces of elliptic curves in the algebraic world
Naïve model for $Y(1)$ as $\mathbb{A}^1_{\mathbb{Q}}$
- Consistent with Galois action on moduli problem

Model for $Y_1(N)$ over $\mathbb{Q}$
- Idea of a fine moduli space
- Outline construction of $Y_1(N)$ as a fine moduli space over $\mathbb{Q}$

Shimura Varieties: Lecture 4

Models of $Y(N)$ over $\mathbb{Q}$ and $\mathbb{Q}(\mu_N)$, Affine algebraic groups

29 October 2014

Martin Orr, Andrei Yafaev

Model for $Y(N)$ over $\mathbb{Q}(\mu_N)$
$Y(N)_{\mathbb{Q}}$ and its connected components over $\mathbb{C}$

Affine algebraic groups
- Definition as a variety with certain morphisms
- Mentioned functor of points

Hopf algebras

Examples of algebraic groups
- $\mathbb{G}_a$, $\mathbb{G}_m$, GL_n
- D_n (diagonal matrices), T_n (upper triangular matrices), U_n (upper triangular unipotent matrices)
- O_n, SO_n, Sp_{2n}

Basic properties of algebraic groups and homomorphisms
- Kernel and image of a homomorphism
- Connected components

Shimura Varieties: Lecture 5

Algebraic tori

12 November 2014

Andrei Yafaev

Definition of algebraic tori
Characters and cocharacters
The Galois action on characters of a torus
The examples of tori over $\mathbb{R}$: $\mathbb{G}_{m,\mathbb{R}}$, $\mathbb{S}^1$ and $\mathbb{S}$

Shimura Varieties: Lecture 6

Reductive groups, Hodge structures

19 November 2014

Andrei Yafaev

Unipotent groups
Semisimple groups
    Defined as almost direct product of simple groups
Reductive groups
    Defined via unipotent radical
    Centre, derived group, adjoint group

Hodge structures
    Definition of $\mathbb{Q}$-HS, $\mathbb{R}$-HS, $\mathbb{Z}$-HS
    Complex tori and HS of type (-1,0), (0,-1)
    Equivalence with representations of $\mathbb{S}$
    Morphisms, direct sums, duals, tensor products of Hodge structures
The Tate Hodge structure $\mathbb{Q}(n)$
Polarisations of Hodge structures

Shimura Varieties: Lecture 6

Mumford–Tate groups, adèles

26 November 2014

Andrei Yafaev

Definition of the Mumford–Tate group of a Hodge structure
MT-invariant subspaces of tensor constructions
Examples of elliptic curves and $\mathbb{Q}(n)$

Absolute values on number fields (normalised so that product formula holds)
Completions of number fields, rings of integers
Restricted direct product of topological groups
Definition of adèles