UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 8

Due: March 10, 2017

1 Let X be a proper variety over an algebraically closed field k. Prove that $\mathcal{O}_X(X) = k$. *Hint: Identify regular functions with morphisms to the affine line.* Which assumptions on X did you actually use in your proof?

2 Let X be a scheme, G a finite group which acts faithfully on X (in other words, $G \subset \mathrm{Aut}(X)$ is a finite subgroup). A *quotient of X by G* is a scheme denoted by X/G together with a morphism $\pi : X \to X/G$ such that for every scheme Y, the induced map

$$\hom(X/G, Y) \xrightarrow{\circ\pi} \hom(X, Y)^G$$

is a bijection. (Of course, the right hand side denotes the set of $f : X \to Y$ such that $f \circ \sigma = f$ for all $\sigma \in G$.)

(a) Let A be a ring on which the finite group G acts faithfully, and denote by $A^G \subset A$ the subring of G-invariants, $\pi : X = \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ the induced morphism of affine schemes. Establish the following:

i. G acts faithfully on X by $\sigma \mapsto \mathrm{Spec}(\sigma^{-1})$.

ii. Given two points $x, y \in X$, show that they lie in the same G-orbit if and only if $\pi(x) = \pi(y)$.

iii. A is an integral extension of A^G.

iv. π is a quotient map (on the underlying topological spaces).

v. $\pi : X = \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ is a quotient of X by G.

(b) (*corrected:*) Back to the general situation, prove that X/G exists if every $x \in X$ admits an affine open neighborhood stable under G.

(c) Assume that X/G exists and that X is separated. Show that X/G is separated.