Homework 4

Due: February 10, 2017

1 Let $(X, \mathcal{O}_X)$ be a ringed space. We are going to associate to it a locally ringed space $(X^l, \mathcal{O}_{X^l})$ as follows.[1] A point of X^l is a pair $(x, \mathfrak{p})$ where $x \in X$ and $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_{X,x})$. Now, let $U \subset X$ be an open subset and $f \in \mathcal{O}_X(U)$ a section. Define the associated *distinguished open subset* to be

$$D(U, f) = \{(x, \mathfrak{p}) \mid x \in U, f_x \notin \mathfrak{p}\} \subset X^l,$$

where f_x denotes the germ of f at x.

(a) Show that the distinguished open subsets are closed under (finite) intersections. We let these be a base for the topology on X^l. Show that the canonical map $\pi : X^l \to X$ which sends $(x, \mathfrak{p})$ to x is continuous.

(b) For $V \subset X^l$ an open subset, we define $\mathcal{O}_{X^l}(V)$ to be the families

$$(s(x, \mathfrak{p}))_{x,\mathfrak{p}} \in \prod_{(x,\mathfrak{p})\in V} (\mathcal{O}_{X,x})_{\mathfrak{p}}$$

such that every $(x, \mathfrak{p}) \in V$ has an open neighborhood $D(U, f) \subset V$ with a section $a/f^n \in \mathcal{O}_X(U)[1/f]$ whose germs satisfy

$$s(y, \mathfrak{q}) = a_y/f_y^n$$

for all $(y, \mathfrak{q}) \in D(U, f)$. Show that this defines a sheaf of rings $\mathcal{O}_{X^l}$ on X^l.

(c) Lift $\pi : X^l \to X$ to a morphism of ringed spaces $(\pi, \pi^\sharp) : (X^l, \mathcal{O}_{X^l}) \to (X, \mathcal{O}_X)$.

(d) Verify that for $(x, \mathfrak{p}) \in X^l$ the induced morphism on stalks

$$\pi_{(x,\mathfrak{p})} : \mathcal{O}_{X,x} \to \mathcal{O}_{X^l,(x,\mathfrak{p})}$$

is precisely the localization of $\mathcal{O}_{X,x}$ at $\mathfrak{p}$. In particular, $(X^l, \mathcal{O}_{X^l})$ is a locally ringed space.

(e) Prove that $(X^l, \mathcal{O}_{X^l})$ is not a scheme in general.

2 We will continue studying the construction in problem 1. Suppose $(\varphi, \varphi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces.

(a) Define $\varphi^l : X^l \to Y^l$ by $(x, \mathfrak{p}) \mapsto (\varphi(x), \varphi_x^{-1}(\mathfrak{p}))$. Prove that $(\varphi^l)^{-1}(D(U, f)) = D(\varphi^{-1}(U), \varphi^\sharp(f))$ and deduce that φ^l is continuous.

(b) Lift φ^l to a morphism of locally ringed spaces $(\varphi^l, \varphi^{l,\sharp}) : (X^l, \mathcal{O}_{X^l}) \to (Y, \mathcal{O}_{Y^l})$. Deduce that we constructed a functor $\mathcal{RS} \to \mathcal{LRS}$ from ringed spaces to locally ringed spaces.

[1] The construction will generalize the construction in class of $\mathrm{Spec}(R)$ for R a ring.

(c) Check that the square

$$\begin{array}{ccc}
(X^l, \mathcal{O}_{X^l}) & \xrightarrow{(\varphi^l, \varphi^{\sharp, l})} & (Y^l, \mathcal{O}_{Y^l}) \\
{\scriptstyle (\pi_X, \pi_X^l)} \downarrow & & \downarrow {\scriptstyle (\pi_Y, \pi_Y^l)} \\
(X, \mathcal{O}_X) & \xrightarrow{(\varphi, \varphi^\sharp)} & (Y, \mathcal{O}_Y)
\end{array}$$

is commutative.

(d) Prove that this construction is *universal*, i.e. for every ringed space $(Y, \mathcal{O}_Y)$ and every morphism of ringed spaces $(\varphi, \varphi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $(X, \mathcal{O}_X)$ is a locally ringed space, there is a unique morphism of locally ringed spaces completing the diagram

$$\begin{array}{ccc}
(X, \mathcal{O}_X) & \xrightarrow{(\varphi, \varphi^\sharp)} & (Y, \mathcal{O}_Y) \\
& \searrow & \uparrow {\scriptstyle (\pi, \pi^\sharp)} \\
& & (Y^l, \mathcal{O}_{Y^l})
\end{array}$$

In other words, there is an adjunction

$$\text{forget} : \mathcal{LRS} \rightleftarrows \mathcal{RS} : (\bullet)^l$$