Homework 3

Due: February 3, 2017

1 In class we defined the tangent space at p, T_pX, of a variety X. It is natural to want to assemble all these spaces (for varying $p \in X$) together into the tangent bundle $\mathrm{T}X$ of X. More precisely, $\mathrm{T}X$ should come with a morphism $\pi : \mathrm{T}X \to X$ whose fibers are the tangent spaces: $\pi^{-1}(p) = \mathrm{T}_pX$. We want to explore to what extent this construction can be performed inside the category of varieties.

(a) As a warm-up, define $\mathrm{T}\mathbb{A}^n$. Notice that it is indeed a variety.

(b) How would you define $\mathrm{T}X$ when X is an affine variety? Is $\mathrm{T}X$ a variety? What is a necessary condition on X for this possibly to be true?

(c) How would you approach the case of arbitrary varieties? What are the problems in carrying out your idea?

2 Recall that a function $f : U \to \mathbb{C}$ on an open subset $U \subset \mathbb{C}$ is analytic if for every $z_0 \in U$ there exists $\varepsilon > 0$ and $f_n \in \mathbb{C}$ such that

$$f(z) = \Sigma_{n\geq 0} f_n(z - z_0)^n, \qquad |z - z_0| < \varepsilon,$$

where $\Sigma_{n\geq 0} f_n\varepsilon^n$ converges absolutely. f is invertible (i.e. $1/f$ is analytic) if and only if $f(z) \neq 0$ for all $z \in U$.

(a) Define, for every open subset $U \subset \mathbb{C}$, $\mathcal{C}^\omega(U)$ to be the set of analytic functions on U. Show that, with the obvious restriction morphisms, this yields a sheaf of $\mathbb{C}$-algebras.

(b) Prove that the stalks of $\mathcal{C}^\omega$ are local rings.

(c) For $U \subset \mathbb{C}$ open define $\mathcal{C}^{\omega,\times}(U) \subset \mathcal{C}^\omega(U)$ to be the set of invertible analytic functions. Show that $\mathcal{C}^{\omega,\times}$ is a sheaf of abelian groups and that the exponential function defines a morphism of sheaves of abelian groups $\exp : \mathcal{C}^\omega \to \mathcal{C}^{\omega,\times}$.

(d) Show that the induced morphism $\exp_{z_0}$ on stalks is surjective for all $z_0 \in \mathbb{C}$.

(e) Show that $\exp_U : \mathcal{C}^\omega(U) \to \mathcal{C}^{\omega,\times}(U)$ is not surjective in general.

3 Let X be a topological space, and let $\mathcal{B}$ be a base for the topology of X which is closed under intersection. Every presheaf F on X restricts to a "presheaf $F|_\mathcal{B}$ on $\mathcal{B}$" (i.e. a presheaf on the full subcategory of $\mathrm{Top}(X)$ spanned by $\mathcal{B}$).

(a) Formulate when a presheaf on $\mathcal{B}$ should be called a sheaf.

(b) Prove that $(\bullet)|_\mathcal{B} : \mathrm{Sh}(X) \to \mathrm{Sh}(\mathcal{B})$ is an equivalence of categories. (If it isn't then your definition in part (a) is not the right one. . .)