UCLA **Winter 2017**
MATH 214A, Introduction to Algebraic Geometry **Martin Gallauer**

Homework 2

Due: January 27, 2017

1 (a) Consider the canonical map $\mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ which sends $(x_0, \ldots, x_n)$ to $[x_0 : \cdots : x_n]$. Prove that this is a morphism of varieties.

(b) Give an example of a continuous map between varieties which is not a morphism.

2 (a) Show that every morphism $\mathbb{A}^n \to \mathbb{A}^1$ has closed image.

(b) Give an example of a morphism $\mathbb{A}^n \to \mathbb{A}^m$ whose image is not closed.

3 Consider the Veronese embedding $v_{3,1} : \mathbb{P}^1 \to \mathbb{P}^3$ and denote its image by X. As mentioned in class, X is a projective variety.

(a) Find 3 homogeneous polynomials which generate the vanishing ideal $\mathcal{I}(X)$ of X.

(b) Show that $\mathcal{I}(X)$ cannot be generated by 2 homogeneous polynomials.

(c) Let Y be the affine variety obtained by intersecting X with one of the standard opens $\mathbb{A}^3 \cong U_i \subset \mathbb{P}^3$. Show that $\mathcal{I}(Y)$ is generated by 2 polynomials.

4 *Part of this exercise is for you to learn the notion of (co)products in a category if you don't know it already. A good place to do so is Chapter 1 of Ravi Vakil's "Foundations of Algebraic Geometry". In fact, these notes* start *with the definition of a product.*

(a) Let A and B be two affine k-algebras. Show that $A \otimes_k B$ is a coproduct of A and B in the category of affine k-algebras.

(b) Deduce that if $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are affine varieties then $X \times Y \subset \mathbb{A}^{m+n}$ with the induced topology is a product of X and Y in the category of varieties.

5 Recall that a conic (hypersurface) in $\mathbb{P}^n$ is the projective variety in $\mathbb{P}^n$ defined by an irreducible homogeneous polynomial of degree 2. Now, assume that $\mathrm{char}(k) \neq 2$. Show that every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. *Hint: Every symmetric bilinear form has an orthogonal basis.*