# Stable homotopy groups of spheres and motivic homotopy theory

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**Abstract**

Programme for reading seminar at Warwick starting in Winter 2024.

## Introduction

**Goals** Motivic homotopy theory was originally envisioned as a setting in which methods from algebraic topology (specifically, homotopy theory) could be brought to bear on questions in algebraic geometry and number theory. As such it has been highly successful. However, it has become increasingly clear that the direction can also be reversed. Methods and tools from motivic homotopy theory have proven very powerful in classical homotopy theory as well. The goal of this reading seminar is to understand a particularly striking instance of that.

One of the fundamental problems in algebraic topology is the computation of stable homotopy groups of spheres, most of which are still unknown. The most significant advances on this problem in recent years resulted from the realization that some of the central tools used in these computations have a motivic origin. This fact can be phrased in different ways, and one of our goals is to understand the most structured one of these. From a topological point of view, it describes a motivic deformation of stable homotopy theory. From a motivic point of view, it provides a purely topological description of a certain category arising in motivic homotopy theory. Both are very striking.

The other goal is to understand how this close relation between topology and algebraic geometry yields computational power that resulted in significant advances in our knowledge of stable homotopy groups of spheres.

**Rough plan** The programme is designed so that people with different interests and background can follow just parts of the seminar. The first part will explain some of the computations of stable homotopy groups of spheres, taking the motivic origin essentially as a black box. This motivic origin manifests itself as a deformation of sorts and this serves as, ehm, motivation for studying deformations of stable homotopy theory. This is the content of the second part, which covers synthetic spectra. These have already found plenty of other applications, so should be of independent interest as well. The third and last part brings the two strands together and establishes the motivic deformation of stable homotopy theory.

**Prerequisites** The first part takes place in the stable homotopy category and some experience with spectra would be good. Also, spectral sequences will feature prominently and experience with this tool would also be good. Depending on how the audience feels about it, this background material could also be recalled at the start.

For the second part, we will use the language of $\infty$-categories and Higher Algebra. I'd say the former is an essential prerequisite (for example, as covered in last term's course) and for the latter we could have a review session if the audience feels this is necessary. Since the third part involves motivic homotopy theory, some background in algebraic geometry is required. However, this prerequisite is so minimal that I expect everyone to know enough. (And the motivic homotopy category will be introduced in the course of the seminar.)

**Talks** The following list of talks is only tentative. We might want to insert some talks recalling additional background material. Some (most? ;-)) talks might also end up taking longer than one hour in which case we should be flexible. I (=Martin) am always happy to chat in case you run into issues while preparing your talk!

### 0.1 Overview

I'm happy to give an overview talk explaining the motivation and direction in which we're heading.

## 1 Stable homotopy groups of spheres

### 1.1 Introduction

Introduce the problem of computing these groups; describe structural results and give plenty of examples. E.g. [Rav86, § 1.1]

### 1.2 Adams

Recall the (dual) mod-$p$ Steenrod algebra, set up the (classical) Adams spectral sequence [Rav86, § 2, 3] and give sample computations. Take this as an opportunity for getting everyone up to speed on spectral sequences. Explain what the Adams filtration is.

### 1.3 Adams-Novikov

Define the complex cobordism spectrum $MU$, describe its coefficient ring and the co-operations. Same for $BP$. Set up the Adams-Novikov spectral sequence based at $MU$ and $BP$. (This is a special case of the theory in [Rav86, § 2] done in the previous talk.) Give sample computations. [Rav86, § 4] Explain what can be achieved with these methods, see also [Isa19].[1] (The case $p = 2$ is different from $p > 2$.)

[1] That is, how many stable homotopy groups of spheres can be computed with these methods?

### 1.4 Cofiber of $\tau$, first take

Follow [Isa19] explaining how the classical Adams spectral sequence is of motivic origin, see [Isa19, §1.3, 3.0.1/2]. (This already suggests a deformation picture at the level of spectral sequences.) Do the same for the classical Adams-Novikov spectral sequence, involving the cofiber of $\tau$. And how this yields information about the spectral sequence. (The introduction (1.2, 1.3, 1.5) gives a nice overview.) It would be nice to give some sample computations. State what can be achieved with these motivic methods.[1]

### 1.5 Cofiber of $\tau$, second take

The deformation picture suggested above is substantiated in [GWX21], see p. 333ff. Follow [GWX21] explaining how the classical algebraic Novikov spectral sequence is of motivic origin, involving the cofiber of $\tau$. And how this yields information about the spectral sequence. (The introduction gives a nice overview.) It would be nice to give some sample computations, see the introduction again, and the appendix. State what can be achieved with these motivic methods, see also [IWX23].[1]

## 2 Deformations of stable homotopy theory

The remaining goal is to explain how the deformation picture at the level of spectral sequences underlies a motivic deformation of stable homotopy theory. This was first shown in [GWX21] and later generalized in [Pst23]. Indeed, the latter considers deformations of stable homotopy theory in some generality, and we will follow that in this part. In the third part, we will specialize to the motivic deformation of interest.

A general remark directed at speakers: This part is quite heavy in ($\infty$-)category-theory and you will encounter many notions you or the audience might be unfamiliar with (to name a few: derived $\infty$-categories, symmetric monoidal $\infty$-categories and modules, additive $\infty$-sites, t-structures, Hopf algebroids, ...). Make sure to introduce these notions. We will start in section 4 of [Pst23] but you will need to go back to sections 2 and 3 to make sense of the material. Finally, it won't be necessary (I think) to talk about hypercomplete sheaves.

There is substantial overlap between the first two talks. The two speakers should liaise about who discusses what.

### 2.1 Motivated introduction to synthetic spectra I

Recall the deformation picture suggested at the level of spectral sequences and use this as motivation for this part. Synthetic spectra (based on an Adams-type homology theory $E$) are a deformation of $\mathsf{Sp}$ in a sense akin to how the derived category is a deformation of its abelian heart. You can sketch its construction following [Chu]. The focus in this talk is on big picture; the details will be given in subsequent talks.

## 2.2 Motivated introduction to synthetic spectra II

The goal of this talk is to define synthetic spectra and understand their basic properties, while still leaving the details for later talks. Following [Pst23, § 4.1–2], start by defining synthetic spectra and observe the basic properties and structures: stable, presentable and symmetric monoidal (Prop 4.2). Continue by introducing the synthetic analogue $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$, and how the monoidal structure on $\mathrm{Syn}_E$ is characterized by $\nu$ (Lem. 4.4, Rem. 4.5).

Start by introducing the bigraded sphere (Def. 4.6), topological degree and Chow degree (Def. 4.8). Continue by defining $Y$-(co)homology of synthetic spectra (Def. 4.9). Note the general calculations Cor. 4.12, Prop. 4.21. Note Prop. 4.26, and use this to introduce $\tau : \nu S^{-1} \to \Omega(\nu S^0)$. Continue with Prop. 4.28 which gives us that this map does indeed describe the degree to which a synthetic spectrum preserves suspensions. Finish by drawing the picture that connects this $\tau$-map with deformations, namely the generic fiber (Cor. 4.34, Thm. 4.37) and the cofiber (Rem. 4.55).

## 2.3 Spherical sheaves on additive sites

From this talk on, we will go back and start filling in the details from the earlier talks, starting with spherical sheaves following § 2.1, 2.5. Start by defining additive $\infty$-sites (Def. 2.3; make sure everyone knows what a site is), recall $P_\Sigma$ and state the basic properties of spherical (pre)sheaves: Cor. 2.7 and Thm. 2.8 (recognition of spherical sheaves), Cor. 2.9. Spherical sheaves are well-behaved with respect to stabilization (Prop. 2.13) and symmetric monoidal structures (Cor. 2.29, Prop. 2.30).

Introduce Grothendieck abelian categories, compact generators (Def. 2.51), epimorphism Grothendieck pretopology (Def. 2.53) and the projective envelope (Def. 2.55). Use this to recognise Grothendieck abelian categories with a choice of compact generators, as spherical sheaves (Thm. 2.58). By 3.2, an example of where this holds is for comodules over so called Adams Hopf algebroids. Further use this to describe $\mathrm{Comod}_{E_*E}$ as spherical sheaves through the equivalence $\mathrm{Comod}_{E_*E} \simeq \mathrm{Sh}_\Sigma^{Set}(\mathrm{Comod}_{E_*E}^{fp})$, when $E$ is "sufficiently nice" (we will come back to this in the next talk), using Rem. 2.65.

## 2.4 Adams-type homology theories and sheaves

In this talk we will introduce the type of spectra on which synthetic spectra will be based, namely "Adams Type", by following § 3.3. Start by defining Adams-type (Def. 3.13, 3.14) and give basic examples (such as the sphere, Landweber exact homology theories and fields). Following Rem. 3.16, note that if $E$ is Adams type, then $E_*E$ is a flat Hopf algebroid[2] and we get the spectral sequence

$$\mathrm{Ext}^{s,t}_{E_*}(E_*X, E_*) \Rightarrow E^{t-s}X.$$

From Thm 3.2 and 2.58 it follows that $\mathrm{Comod}_{E_*E}$ is a Grothendieck abelian category. Continue by showing that $\mathrm{Sp}_E^{fp}$ is an excellent $\infty$-site (Prop. 3.23) and that $y(X)$ is a spherical sheaf (Prop. 3.24). Using Thm. 3.27 and Thm. 3.2, conclude that if $E$ is Adams-type, then $E_*$ :

[2] Make sure everyone knows what a Hopf algebroid is.

$\mathrm{Sp}_E^{fp} \to \mathrm{Comod}_{E_*E}^{fp}$ induces a symmetric monoidal equivalence

$$\mathrm{Sh}_\Sigma^{set}(\mathrm{Sp}_E^{fp}) \simeq \mathrm{Sh}_\Sigma^{set}(\mathrm{Comod}_{E_*E}^{fp}) \simeq \mathrm{Comod}_{E_*E}.$$

## 2.5 The t-structure on synthetic spectra

We now have all of the preliminary knowledge necessary to start truly understanding synthetic spectra, so start by recalling this definition. Introduce the general notion of t-structures, give examples (derived category, spectra) and introduce the t-structure on spherical sheaves (Def. 2.15, Prop 2.16). Use this to understand the t-structure on synthetic spectra which satisfies

$$\mathrm{Syn}_E^\heartsuit \simeq \mathrm{Sh}_\Sigma^{Ab}(\mathrm{Sp}_E^{fp}) \simeq \mathrm{Sh}_\Sigma^{Set}(\mathrm{Sp}_E^{fp}) \simeq \mathrm{Comod}_{E_*E},$$

as in Prop. 4.16. Give the explicit form of this equivalence just after 4.16. Recall the definition of the synthetic analogue $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$, introduce $\pi_k^\heartsuit X$ for $X \in \mathrm{Syn}_E$ and use this to describe the relation between synthetic homology and t-structure homotopy groups (Thm. 4.18). Conclude with 4.19–21.

## 2.6 The generic fiber

Following § 4.3, start by recalling the thread/deformation map $\tau : \nu S^{-1} \to \Omega(\nu S^0)$ (Def. 4.27) and Prop. 4.28. Describe the connection between $\tau$ and t-structure covers of synthetic analogues (Rem. 4.31 after 4.29 and 4.30 – the cofiber of $\tau$ will be more important in the following talk). Define $\tau$-invertible (Def. 4.32) and discuss Prop. 4.33 and the equivalence $\mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1}S^{0,0}}(\mathrm{Syn}_E)$ (Cor. 4.34). Define the spectral Yoneda embedding, and prove that $\mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1})$ (Thm. 4.37) – this is the generic fiber of the deformation.

## 2.7 The special fiber

The goal of this talk is to understand the modules over the cofiber of $\tau$ in terms of Hovey's stable $\infty$-category of comodules. Start by introducing $\mathrm{Stable}_\Gamma$ as in the beginning of § 3.2, and Thm. 3.7. Note that this is not equivalent to $\mathcal{D}(\mathrm{Comod}_{E_*E})$ in general. Following Cor. 3.8, sketch how the connective lift of the Yoneda embedding gives an equivalence $\Sigma_+^\infty y : \mathrm{Comod}_{E_*E} \xrightarrow{\simeq} \mathrm{Stable}_{E_*E}^\heartsuit$. Next, construct the adjunction in Lem. 4.43 between $\mathrm{Syn}_E$ and $\mathrm{Stable}_{E_*E}$, and prove the equivalence $\mathrm{Stable}_{E_*E}^\heartsuit \simeq \mathrm{Syn}_E^\heartsuit$ from Lem. 4.44 as well as the equivalence from Lem. 4.45. The main result of this talk is Thm. 4.46 which identifies modules over the cofiber of $\tau$, with a subcategory of Hovery's stable $\infty$-category of comodules. Mention (Prop. 4.53, Lem. 4.51) examples of when this inclusion is an equivalence. (No need to introduce "plenty of projectives" nor discuss the proof of 4.53. In our main application, these assumptions are not satisfied.)

## 2.8 Even synthetic spectra based on MU

1 talk covering [Pst23, § 5.2, 6].

# 3 Comparison with stable motivic homotopy theory

The goal of this third part is to establish an equivalence between $p$-completed cellular motivic spectra over Spec($\mathbb{C}$) and even synthetic spectra based on $MU$. This will draw upon both preceding parts.

You can find some of the topics below also covered in the 2023 Talbot programme. See there for references (or ask me).

## 3.1 Motivic homotopy theory

Define $\mathsf{Sp}_k$, the stable motivic homotopy ($\infty$-)category over a field $k$. (Don't follow the notation in [Pst23]: The cellular part should have a separate name.) State basic properties and sample computations, e.g. $\pi_0(\mathbb{S}^0_k)$, motivic cohomology of a point, the dual Steenrod algebra. Construct the Betti realization $\mathsf{Sp}_{\mathbb{C}} \to \mathsf{Sp}$ and describe what it does on objects and maps of interest (e.g. spheres, notable maps between spheres: $\rho$, $\eta$, $\tau$). Define the cellular subcategory and give examples of cellular objects.

## 3.2 MGL

Construct the motivic ring spectrum $MGL$ as a cellular object and give the main results of [Pst23, § 7.1, 7.2].

## 3.3 As spherical sheaves

[Pst23, § 7.3]

## 3.4 Finishing up

[Pst23, § 7.4, 7.5] Also describe the resulting deformation picture again.

# References


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