

Weighted and fractional Poincaré Inequalities

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based on work with

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Classical Poincaré

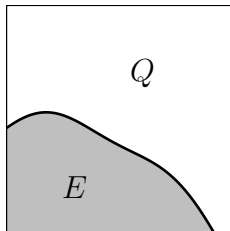
Poincaré inequality

$$\int_Q |f - f_Q| \lesssim_d l(Q) \int_Q |\nabla f|$$

with $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$. It's equivalent to

relative isoperimetric inequality

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{d-1} \lesssim_d \mathcal{H}^{d-1}(Q \cap \partial E)^d$$



For $1 \leq p \leq d$ there exist strengthened versions

Strengthened p -Poincaré

$$\left(\int_Q |f - f_Q|^{p^*} \right)^{\frac{1}{p^*}} \lesssim_d \left(\int_Q |\nabla f|^p \right)^{\frac{1}{p}}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

With $Q \rightarrow \mathbb{R}^n$ we obtain Sobolev embedding from p -Poincaré.

$$\|f\|_{p^*} \lesssim_d \|\nabla f\|_p.$$

Theorem (Bourgain, Brezis, and Mironescu 2002; Maz'ya and Shaposhnikova 2002; Ponce 2004; Milman 2005)

Let $0 \leq \delta < 1$. Then

$$\int_Q |f - f_Q| \lesssim_d (1 - \delta) l(Q)^\delta \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{d+\delta}} dx dy \lesssim l(Q) \int_Q |\nabla f|$$

- Without $(1 - \delta)$ the first inequality follows from the triangle inequality.

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$$\int_Q \frac{1}{r^{d+\delta-1}} \sim_d \frac{l(Q)^{1-\delta}}{1-\delta}$$

- L^p version with $\frac{1}{p_\delta^*} = \frac{1}{p} - \frac{\delta}{d}$.

With weights

For $0 \leq \alpha \leq d$ the fractional maximal function is

$$M_\alpha \mu(x) = \sup_{r>0} r^\alpha \frac{\mu(B(x, r))}{\mathcal{L}(B(x, r))}.$$

Theorem (Franchi, Pérez, and Wheeden 2000)

Let $1 \leq q \leq \frac{d}{d-1}$. Then

$$\left(\int_Q |f - f_Q|^q d\mu \right)^{\frac{1}{q}} \lesssim_{d,q} \int_Q |\nabla f(x)| M_{d-q(d-1)} \mu(x)^{\frac{1}{q}} dx.$$

- Constant blows up for $q \searrow 1$, but is finite for $q = 1$.
- Generalizes Meyers and Ziemer 1977 who consider $\mu(x) \lesssim |x|^{-\alpha}$ which implies $M_\alpha \mu \lesssim 1$.

Theorem (Myryläinen, Pérez, and Weigt 2023)

Let $0 \leq \delta < 1$ and $1 \leq q \leq \frac{d}{d-\delta}$. Then

$$\left(\int_Q |f - f_Q|^q \mu \right)^{\frac{1}{q}} \lesssim_d (1 - \delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{n+\delta}} dy M_{d-q(d-\delta)} \mu(x)^{\frac{1}{q}} dx$$

- Hurri-Syrjänen, Javier C. Martínez-Perales, et al. 2022: case $q = \frac{d}{d-\delta}$.
- Implies Franchi, Pérez, and Wheeden 2000 without blowup at $p \rightarrow 1$.
- $d - q(d - \delta)$ is optimal.

Theorem (Myyryläinen, Pérez, and Weigt 2023)

Let $0 < \delta < 1$. Then

$$(1 - \delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{d+\delta}} dy d\mu(x) \lesssim_d \frac{|(Q)|^{1-\delta}}{\delta} \int_Q |\nabla f| M\mu dx.$$

- Already proven in Hurri-Syrjänen, Javier C Martínez-Perales, et al. 2023 but only with extra factor $1 - \delta$.
- Can replace $M\mu$ by μ if $\mu \in A_1$.
- Since $M\mu \in A_1$ for general μ we can deduce classical weighted Poincaré
- Also have version for $p > 1$.

Theorem (Myrskyläinen, Pérez, and Weigt 2023)

Let $0 < \delta < 1$, $1 \leq r \leq p < \frac{d}{\delta}$ and let q be defined by

$$\frac{1}{p} - \frac{1}{q} = \frac{\delta}{nr}.$$

Then for all cubes $Q \subset \mathbb{R}^d$, $f \in L^1(Q)$ and $w \in A_r$ we have

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \left(\frac{1}{w(Q)} \int_Q |f - c|^q w \, dx \right)^{\frac{1}{q}} \\ & \lesssim q [w]_{A_r}^{\frac{1}{p} + \frac{\delta}{nr} + 1} \frac{(1 - \delta)^{\frac{1}{p}}}{\delta^{1 - \frac{1}{p}}} |Q|^\delta \\ & \quad \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n + \delta p}} \, dy \, w(x) \, dx \right)^{\frac{1}{p}}. \end{aligned}$$

$p > 1$? no, for δ large

Counterexample (Myyryläinen, Pérez, and Weigt 2023)

For any $1 < p < n$, $p \leq q \leq \frac{np}{n-p}$, $\alpha = n - \frac{q}{p}(n-p)$, and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f with

$$\left(\int_Q |f - f_Q|^q d\mu \right)^{\frac{1}{q}} > C \left(\int_Q |\nabla f|^p (M_{\alpha}\mu)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

- Is also counterexample against weighted fractional p -Poincaré for δ near 1.
- What about $\delta = 0$?

$p > 1$? yes, with ε -loss

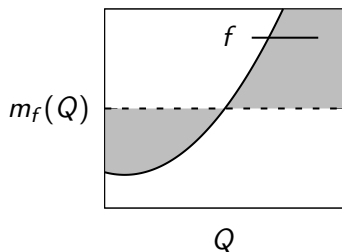
Theorem (Hurri-Syrjänen, Javier C. Martínez-Perales, et al. 2022)

$$\left(\int_Q |f - f_Q|^p \right)^{\frac{1}{p}} \lesssim_d (1 - \delta)^{\frac{1}{p}} \frac{|(Q)|^\varepsilon}{\varepsilon} \left(\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{d + \delta p}} dx M_{(\delta - \varepsilon)p} \mu(y) dy \right)^{\frac{1}{p}}$$

- Not optimal for $p = 1$ by our result.
- Is there a unified result for all p ?

Classical Poincaré by isoperimetric inequality

Consider median $m_f(Q)$ instead of the average



$$\blacksquare = \int_Q |f - m_f(Q)| = \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\}) d\lambda + \int_{-\infty}^{m_f(Q)} \mathcal{L}(\{x \in Q : f(x) < \lambda\}) d\lambda$$

Classical Poincaré by isoperimetric inequality

$$\begin{aligned} & \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\}) \, d\lambda \\ & \leq I(Q) \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})^{\frac{d-1}{d}} \, d\lambda \\ & \lesssim I(Q) \int_{m_f(Q)}^{\infty} \mathcal{H}^{d-1}(\partial\{x \in Q : f(x) > \lambda\}) \, d\lambda \\ & \leq I(Q) \int_Q |\nabla f| \end{aligned}$$



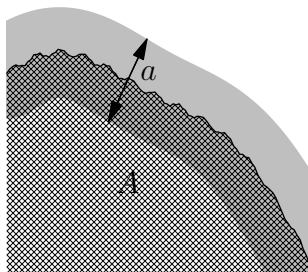
general measure: replace $\mathcal{L} \rightarrow \mu$, weigh \mathcal{H}^{d-1} with $M_\alpha \mu$.

Fractional Poincaré

Lemma (Fractional relative isoperimetric inequality)

Let $a > 0$ and $A \subset Q$ with $a^d \leq \mathcal{L}(A) \leq \mathcal{L}(Q)/2$. Then

$$\begin{aligned} a\mathcal{L}(Q \cap A)^{\frac{d-1}{d}} &\lesssim \int_Q \int_{Q \cap B(x,a) \setminus B(x,a/2)} |1_A(x) - 1_A(y)| \, dy \, dx \\ &\lesssim a\mathcal{H}^{d-1}(Q \cap \partial A) \end{aligned}$$



Theorem (Myrskyläinen, Pérez, and Weigt 2023)

$$\left(\int_Q |f - f_Q|^q \mu \right)^{\frac{1}{q}} \lesssim_d (1 - \delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{n+\delta}} dy M_{d-q(d-\delta)} \mu(x)^{\frac{1}{q}} dx$$

- 1 write LHS in terms of levelsets
- 2 insert geometric sum over $\sum_{k \geq 0} 2^{-k(1-\delta)} \sim \frac{1}{1-\delta}$
- 3 apply fractional relative isoperimetric inequality with $a \sim 2^{-k} l(Q)$
- 4 evaluate integral over levelsets to recover difference quotient with $|x - y| \sim a \sim 2^{-k} l(Q)$, using Fubini and

$$\int_{\mathbb{R}} |1_{\{f > \lambda\}}(x) - 1_{\{f > \lambda\}}(y)| d\lambda = |f(x) - f(y)|.$$



Thank you