Higher Dimensional Techniques for the Regularity of Maximal Functions

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18.05.2023

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \qquad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

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Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\partial_e \mathrm{M}^{\mathrm{c}} f(x) \sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h}$$

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&= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x)
\end{aligned}$$

Introduction: Background

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Proof: For $e \in \mathbb{R}^n$ by the sublinearity of $\mathrm{M^c}$

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By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathrm{M^c}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \leq C_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Question (Hajłasz and Onninen 2004)

Is it true that

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Uncentered Hardy-Littlewood maximal function

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for $\widetilde{\mathrm{M}}$ and other maximal operators.

Introduction: In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For $f : \mathbb{R} \to \mathbb{R}$ *we have*

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq \|\nabla f\|_1$$

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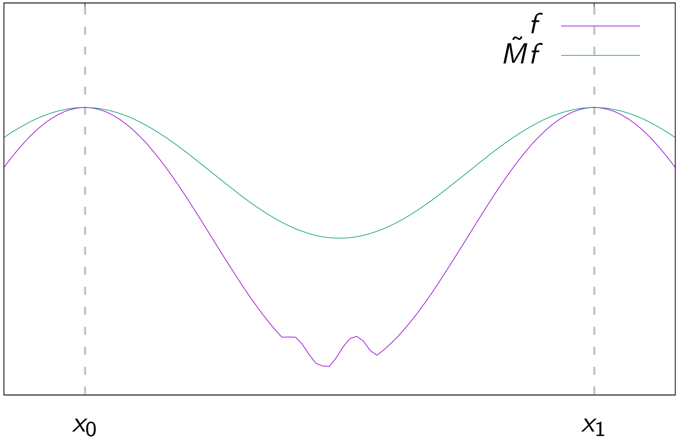
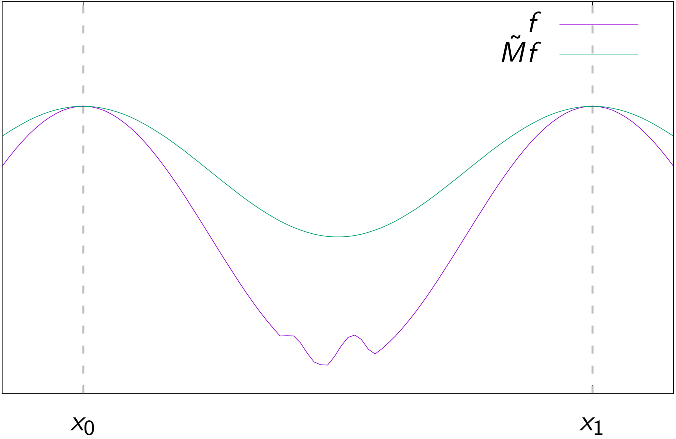
In one dimension:

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \ldots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f$$

and $\widetilde{\mathrm{M}} f$ is convex on connected components of $\{x \in \mathbb{R} : \widetilde{\mathrm{M}} f(x) > f(x)\}$.



f
$\tilde{M}f$
x_0
x_1



$$\operatorname{var}_{[x_0,x_1]} \widetilde{\mathrm{M}} f \leq \operatorname{var}_{[x_0,x_1]} f$$

For $0 < \alpha < n$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}^{\mathrm{c}}_{\alpha} f(x) = \sup_{r>0} r^{\alpha} f_{B(x,r)}.$$

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Corresponding Hardy-Littlewood theorem

$$\|\mathrm{M}_{\alpha} f\|_{L^{p_\alpha}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

with $p_\alpha = \frac{pn}{n-\alpha p} > p$ if and only if $p > 1$.

Introduction: The fractional maximal function

For $0 < \alpha < n$ the centered fractional Hardy-Littlewood maximal function is

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Corresponding Hardy-Littlewood theorem

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with $p_\alpha = \frac{pn}{n-\alpha p} > p$ if and only if $p > 1$. Corresponding regularity bound

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{p_\alpha}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

proven for $p > 1$.

Theorem (Kinnunen and Saksman 2003)

For $\alpha \geq 1$

$$|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f(x)| \lesssim |\mathrm{M}^{\mathrm{c}}_{\alpha-1} f(x)|.$$

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For $\alpha \geq 1$ we have $1_\alpha = \left(\frac{n}{n-1}\right)_{\alpha-1}$ and therefore

$$\|\nabla \mathrm{M}_\alpha^{\mathrm{c}} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim \|\mathrm{M}_{\alpha-1}^{\mathrm{c}} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$$
$$\lesssim \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

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- What about $0 < \alpha < 1$?

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- What about $0 < \alpha < 1$?
- Same result for $\widetilde{\mathrm{M}}_\alpha$.

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block decreasing f	[Aldaz+Pérez Lázaro 2009]
centered M, $n = 1$	[Kurka 2015]
radial f	[Luiro 2018]

Introduction: Past progress

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more related bounds, bounds on other maximal operators, such as local,. . . , for example: Continuity of $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, stronger than boundedness.

We prove the endpoint regularity bound for the maximal function for

- characteristic f

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- dyadic maximal operator

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- fractional maximal operator
- cube maximal operator

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

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Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

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Introduction: Proof ingredients

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Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1}\mathcal{L}(B)\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly.

Introduction: Proof ingredients

1. **relative isoperimetric inequality**:

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

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	isoperimetric, Vitali	boundary Besicovitch	superlevel
dyadic char. f.	**x**		
char. f.	**x**	**x**	
dyadic	**x**		**x**
fractional	**x**		**x**
cube	**x**	**x**	**x**

Proof: Reformulation and decomposition

We have

$$\{\mathrm{M}f > \lambda\} = \bigcup \mathcal{B}_\lambda^{<} \cup \bigcup \mathcal{B}_\lambda^{\geq}.$$

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Since $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ we have

$$\partial\{\mathrm{M}f > \lambda\} \subset (\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) \cup \partial\{f > \lambda\}.$$

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We conclude

Decomposition

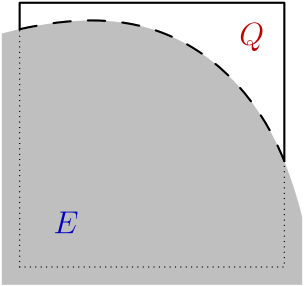
$$\int_{\mathbb{R}^d} |\nabla \mathrm{M}f| \leq \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^\geq \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda$$

Proof: High density case $\mathcal{B}_\lambda^{\geq}$

Proposition

For Q, E with $\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q)$ we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) \lesssim \mathcal{H}^{n-1}(Q \cap \partial E)$$



dyadic maximal operator

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&\lesssim \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}\big(Q \cap \partial \{f > \lambda\}\big) \\
&\leq \mathcal{H}^{n-1}\big(\partial \{f > \lambda\}\big)
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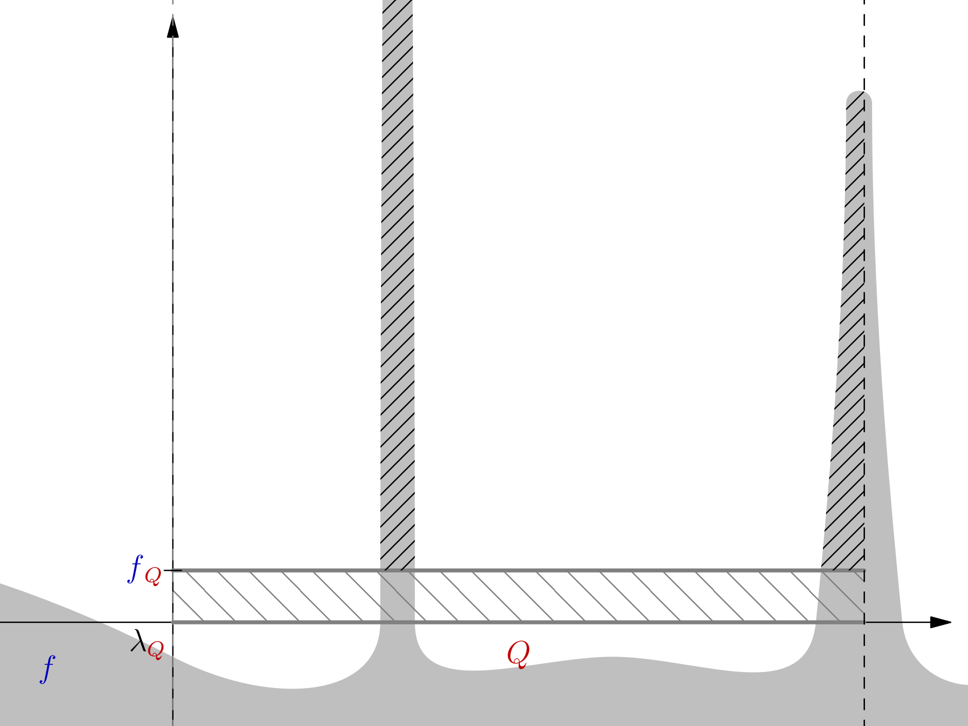

f_Q
λ_Q
f
Q

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \le \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q)\mathcal{H}^{n-1}(\partial Q)$$

with

$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1}\mathcal{L}(Q)$$

Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim \int_{f_Q}^{\infty} \mathcal{L}(Q \cap \{f > \lambda\}) \qquad \mathrm{d}\lambda$$

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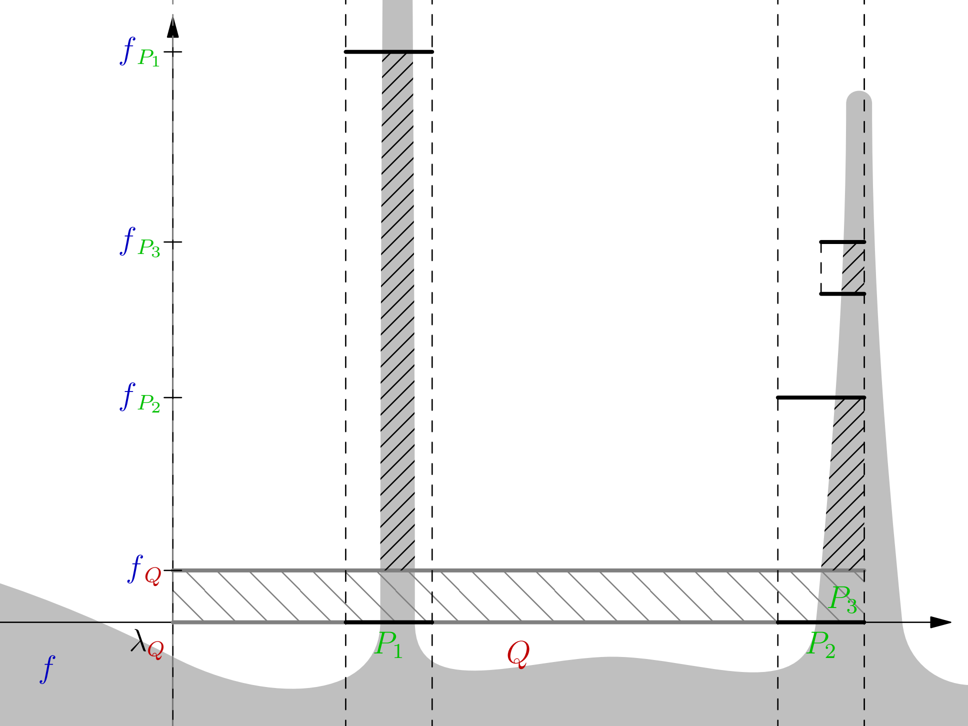
where P is maximal above $\bar{\lambda}_P$ and

$$\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1}\mathcal{L}(P)$$



f_{P_1}
f_{P_3}
f_{P_2}
f_Q
λ_Q
f
P_1
Q
P_2
P_3

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

Combining, we obtain

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda$$
$$\lesssim \int_{\mathbb{R}} \sum_{Q \text{ dyadic}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(Q)} \, \mathrm{d}\lambda$$

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Combining, we obtain

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1. change the order of summation
2. convergence of the geometric sum
3. apply the relative isoperimetric inequality to P.
4. coarea formula to recover $\|\nabla f\|_1$

Proof: Low density case $\mathcal{B}_\lambda^<$, fractional

$1 \leq \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|f\|_{\frac{n}{n-1}} \lesssim \|\nabla f\|_1.$$

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$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha,-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|\nabla f\|_1,$$

$\mathrm{M}_{\alpha,-1}$ replacement for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$.

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Can bound $\mathrm{M}_{\alpha,-1} f$ both centered and uncentered

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$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha,-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|\nabla f\|_1,$$

$\mathrm{M}_{\alpha,-1}$ replacement for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$.
Can bound $\mathrm{M}_{\alpha,-1} f$ both centered and uncentered

- using low density arguments from the dyadic proof

Proof: Low density case $\mathcal{B}_\lambda^<$, fractional

$1 \le \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|f\|_{\frac{n}{n-1}} \lesssim \|\nabla f\|_1.$$

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$\mathrm{M}_{\alpha,-1}$ replacement for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$.
Can bound $\mathrm{M}_{\alpha,-1} f$ both centered and uncentered

- using low density arguments from the dyadic proof
- extra flexibility coming from $\alpha > 0$, allowing for rough Vitali covering arguments

Proof: Low density case $\mathcal{B}_\lambda^<$, general cubes

cube maximal function

$$\mathrm{M}f(x) = \sup_{\text{cube } Q,\ Q \ni x} f_Q .$$

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We reduce to almost dyadic cubes, using

Proposition (Vitali/Besicovitch for perimeter)

For any (finite) set of cubes $\mathcal{Q}$ there is a subset $\mathcal{S} \subset \mathcal{Q}$ of disjoint cubes such that

$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{Q}\Big) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{n-1}(\partial S).$$

Uncentered HL $\widetilde{\mathrm{M}}f$ (balls)?

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- except low density bound $(f_B - \lambda_B)\mathcal{L}(B) \lesssim$?

Thank you