The variation of maximal functions in higher dimensions

Lectio Praecursoria

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Aalto University

21.10.2022

The *variation* of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$\operatorname{var} f = |f(x_1) - f(x_2)| + |f(x_2) - f(x_3)| + \ldots + |f(x_{N-1}) - f(x_N)|$$

if $x_1 < x_2 < \ldots < x_N$ are the local minima and maxima of f.

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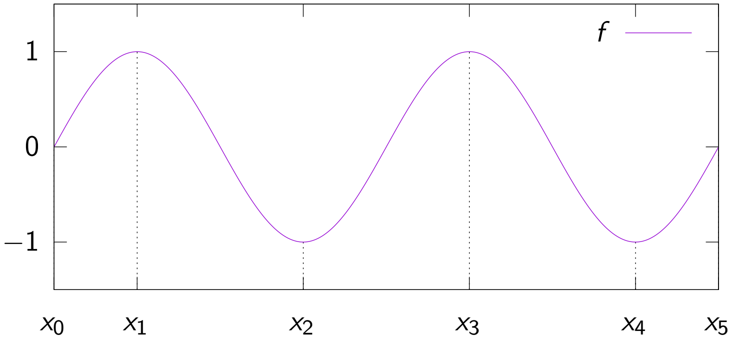

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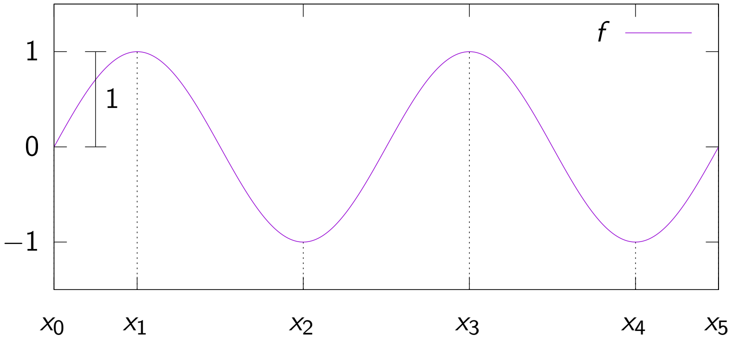

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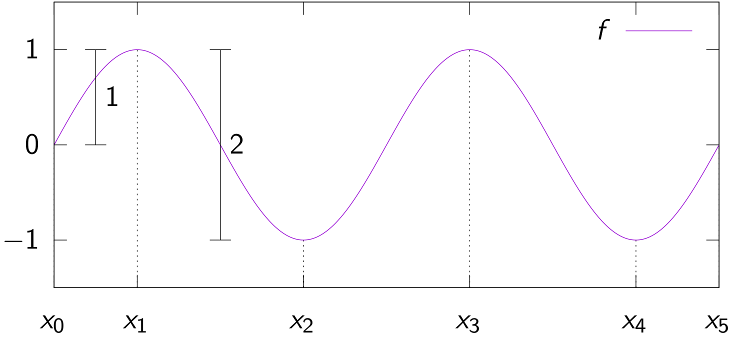

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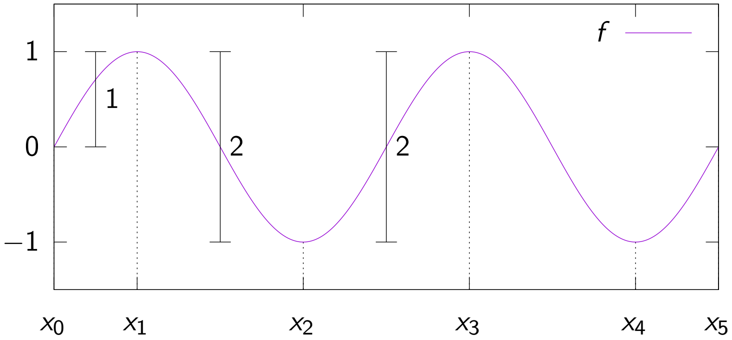

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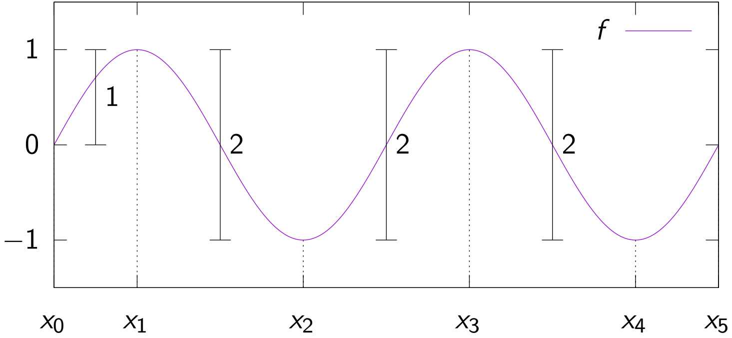

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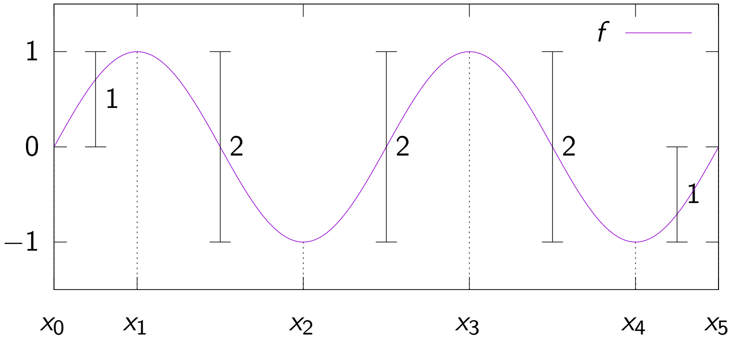

$$\operatorname{var} f = 1 + 2 + 2 + 2 + 1 = 8$$

Regularity

- The variation is one measure of *regularity*.

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- smoothness of a function, how nice does a function look
- regularity of solutions to partial differential equations

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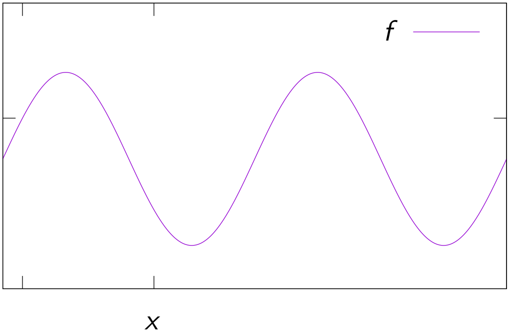

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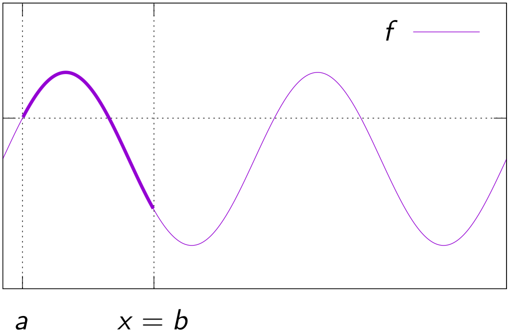

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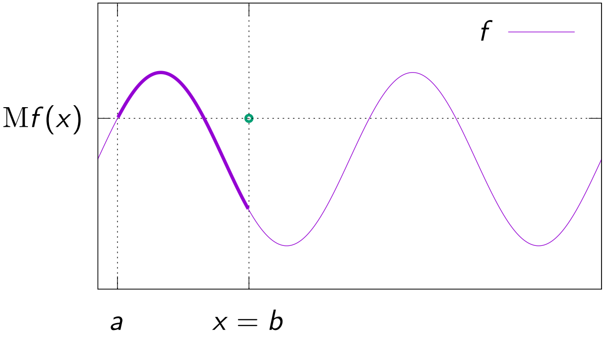

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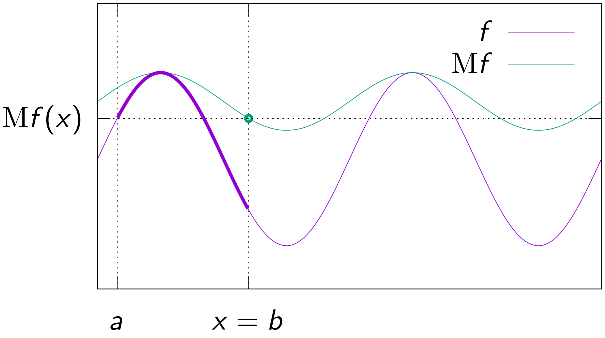

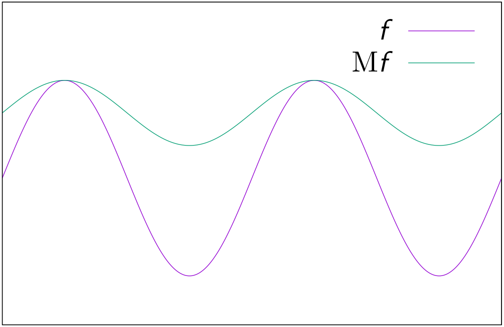
f
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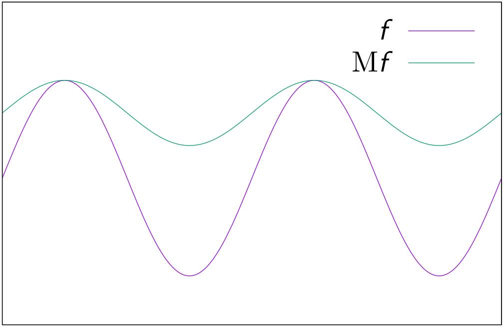

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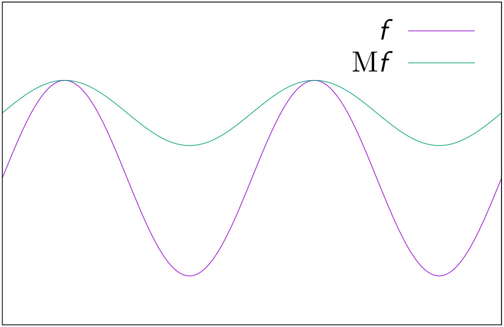

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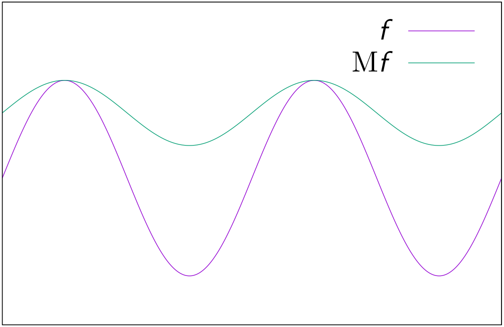

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$d \geq 2$?

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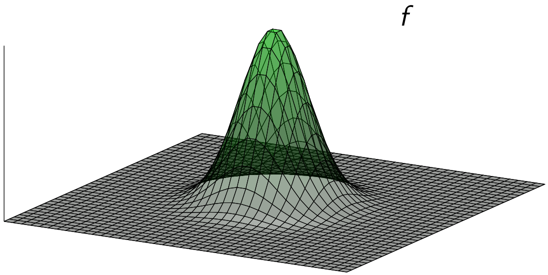

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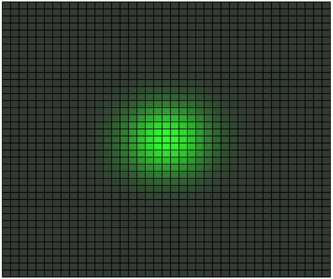

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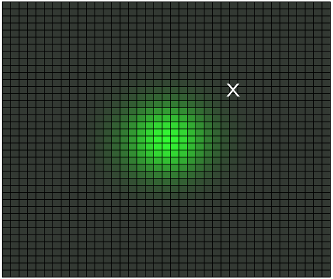

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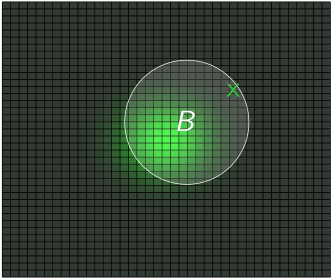

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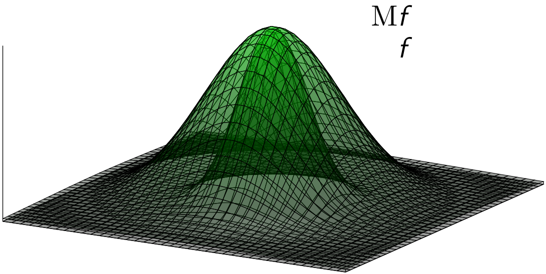

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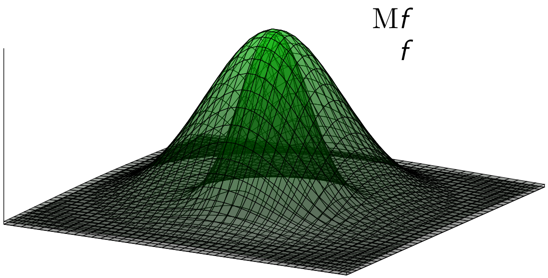

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- continuity of the gradient (harder) [Carneiro, González-Riquelme, Kosz, Luiro, Madrid, Nuutinen, Pierce,. . .]

Results in this thesis

$$\operatorname{var} \mathrm{M}f \leq C \operatorname{var} f \quad ?$$

1. characteristic functions

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Techniques

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4. dyadic decompositions

Coarea formula and superlevelsets

Coarea formula

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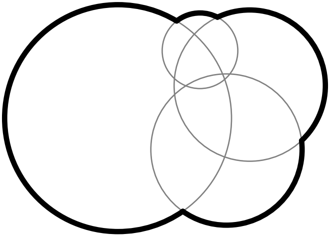

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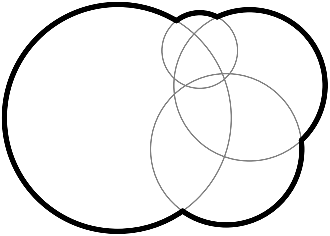


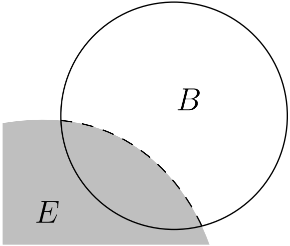
- the starting point in all of the articles

Geometric estimates

Relative isoperimetric inequality

For any ball B and set E with $\mathcal{L}(B \cap E) \leq \frac{1}{2}\mathcal{L}(B)$ we have

$$\mathcal{L}(B \cap E)^{d-1} \leq C \cdot \mathcal{H}^{d-1}(B \cap \partial E)^d$$



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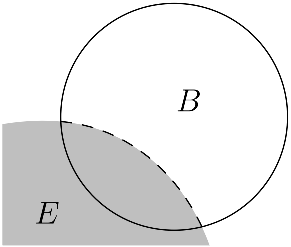

- classical result
- used extensively

Geometric estimates

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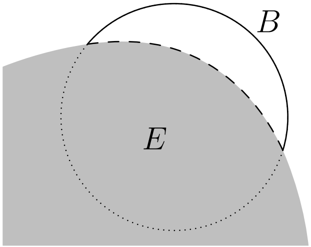

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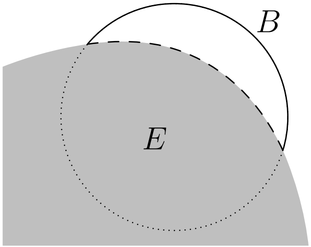

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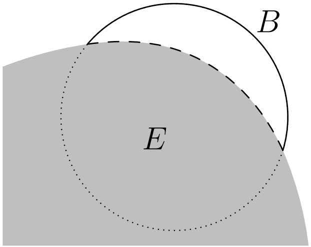

- proved in first publication
- used in most following publications

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Proposition

Let B be a ball and $\mathcal{B}$ be a set of balls C with $r(C) \geq r(B)$. Then

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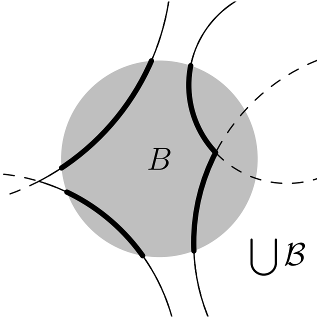

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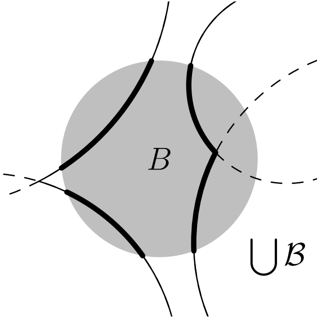

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Let $\mathcal{B}$ be a set of balls. Then it has a disjoint subcover $\mathcal{B}^*$ with

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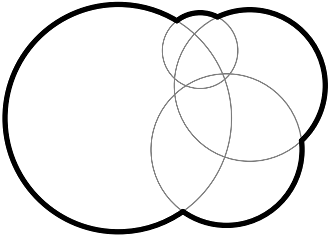

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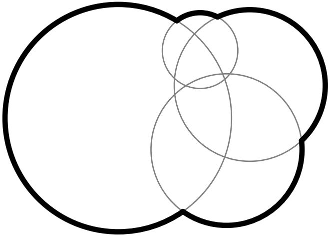

- first ideas developed in first publication

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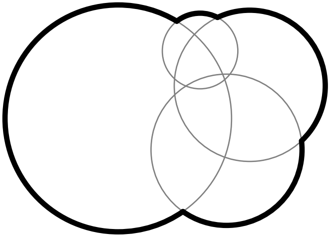

- first ideas developed in first publication
- similar result proved in last publication

Dyadic decomposition

- way to decompose a function into parts with respect to the local scale of its variation

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- way to decompose a function into parts with respect to the local scale of its variation
- key development in second publication
- main ingredient in third publication

Dyadic decomposition

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- fifth publication combines geometric estimates and dyadic decomposition