Almost-Orthogonality of Restricted Haar Functions

Master's thesis talk

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Kadison-Singer

V is a

Bessel sequence if $$\|\sum_{v\in V} a_v v\|^2 \leq C \sum_{v\in V} \|a_v v\|^2$$

Riesz basic sequence if also $$\|\sum_{v\in V} a_v v\|^2 \geq c \sum_{v\in V} \|a_v v\|^2$$

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Theorem (Marcus, Spielman, Srivastava 2014)

Let V be a Bessel sequence with $C < \frac{4}{3}$. Then V can be partitioned into two Riesz basic sequences.

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$\mathcal{D}$: dyadic intervals in $[0,1]$. For $I \in \mathcal{D}$ define a *Haar function* by

$$h_I(x) := \begin{cases} -1 & x \in I_{\mathfrak{l}} \\ 1 & x \in I_{\mathfrak{r}} \end{cases}$$

$\{h_I \mid I \in \mathcal{D}\}$ is an orthogonal subset of $L^2([0,1])$.

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Corollary

Let $p > \frac{3}{4}$ and $E \subset [0,1]$. Then

$$\{h_I \mathbb{1}_E \mid I \in \mathcal{D},\ |I \cap E| \geq p|I|\}$$

can be partitioned into two Riesz basic sequences.

Main result

Theorem

$p > \frac{2}{3}$ if and only if for all $E \subset [0,1]$

$$\left\{ h_I \mathbb{1}_E \;\middle|\; I \in \mathcal{D},\ |I \cap E| \geq p|I| \right\}$$

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is a Riesz basic sequence.

$$\Big\| \sum_{I \in \mathcal{D}} a_I h_I \mathbb{1}_E \Big\|_2^2 \geq c \sum_{I \in \mathcal{D}} \| a_I h_I \mathbb{1}_E \|_2^2$$

where $a_I = 0$ if $|I \cap E| < p|I|$.

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Theorem

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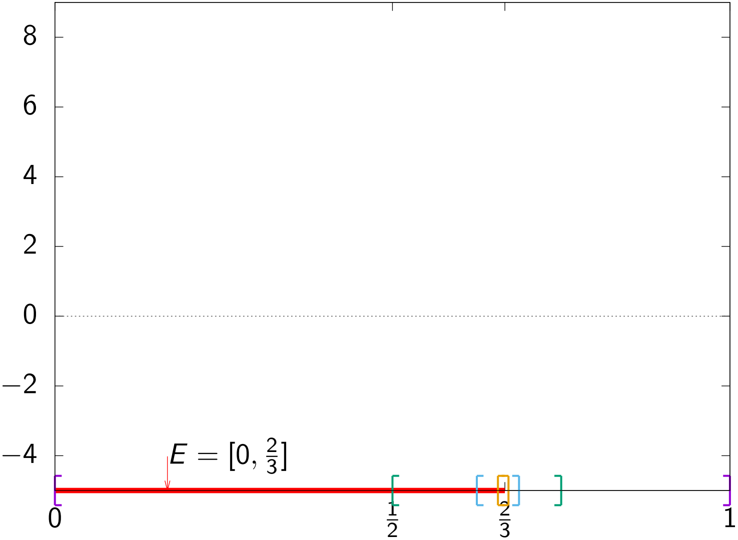
Remark

If $E = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$ then $h_{[0,1]} \mathbb{1}_E - h_{[0,\frac{1}{2}]} \mathbb{1}_E + h_{[\frac{1}{2},1]} \mathbb{1}_E = 0$ so $p > \frac{1}{2}$ is necessary. $p = 1$ is sufficient.

Counterexample for $p = \frac{2}{3}$



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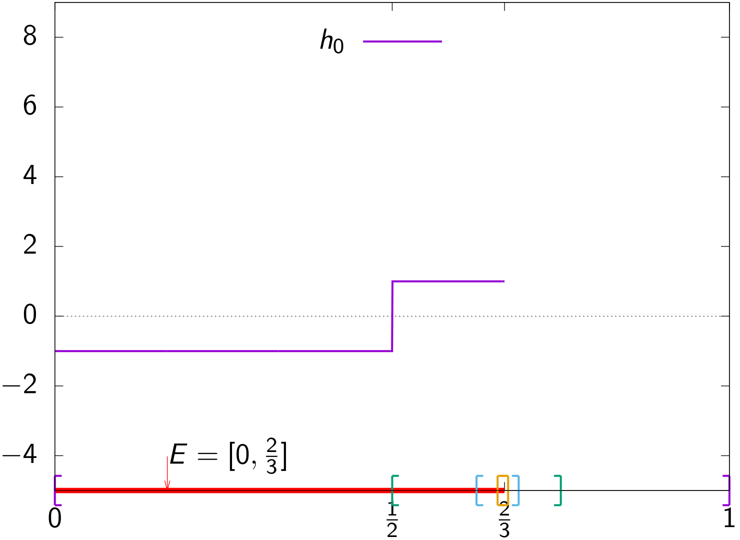

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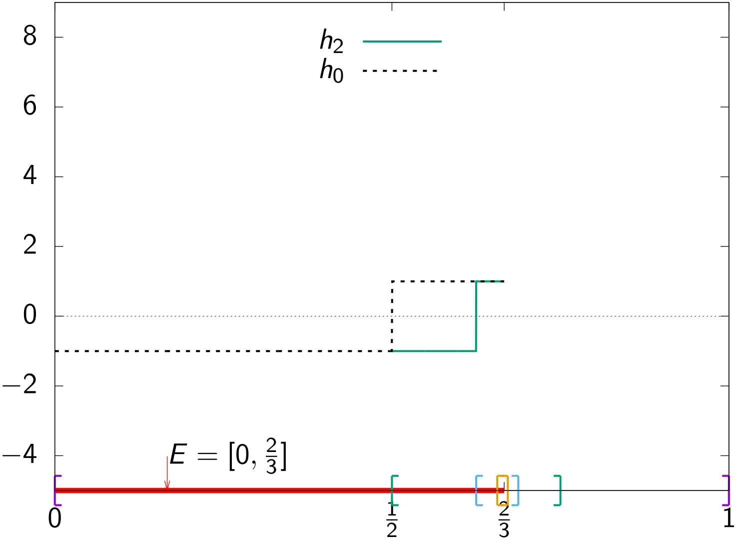

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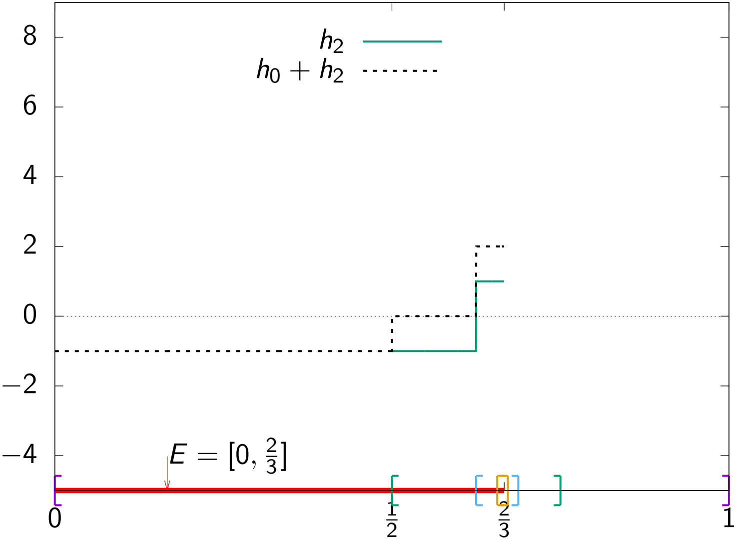

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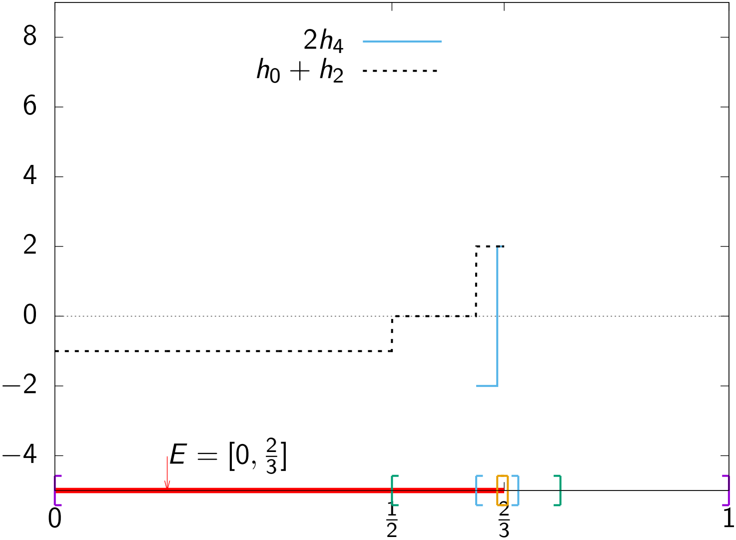

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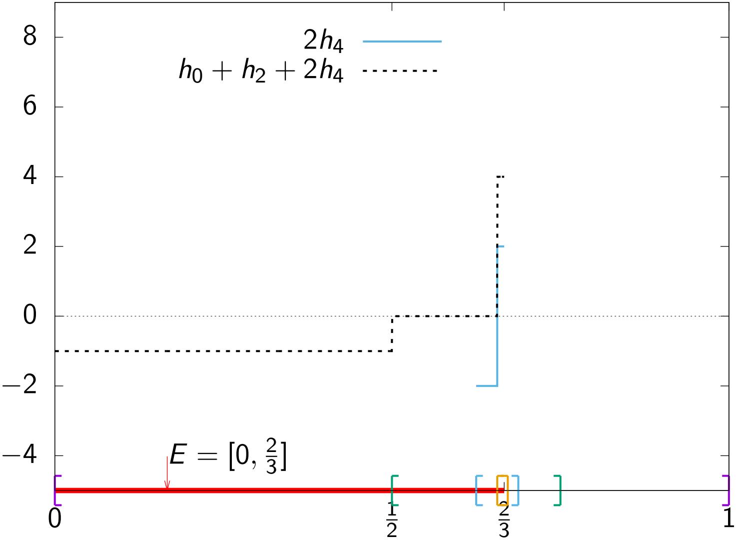

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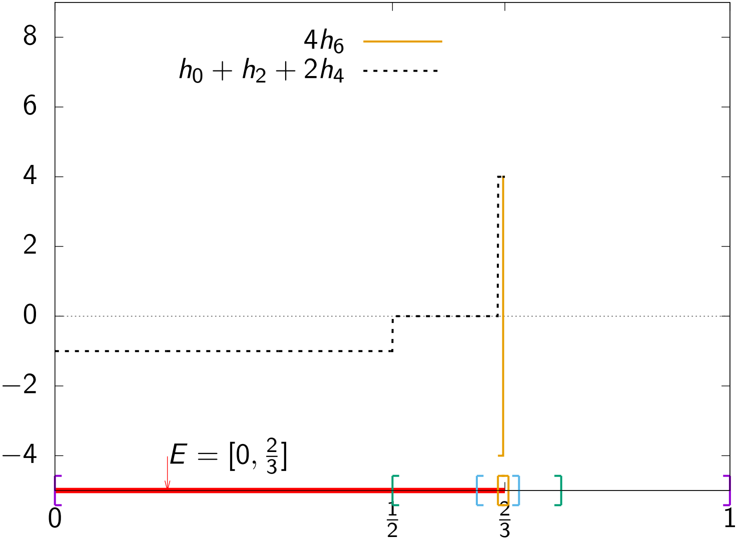

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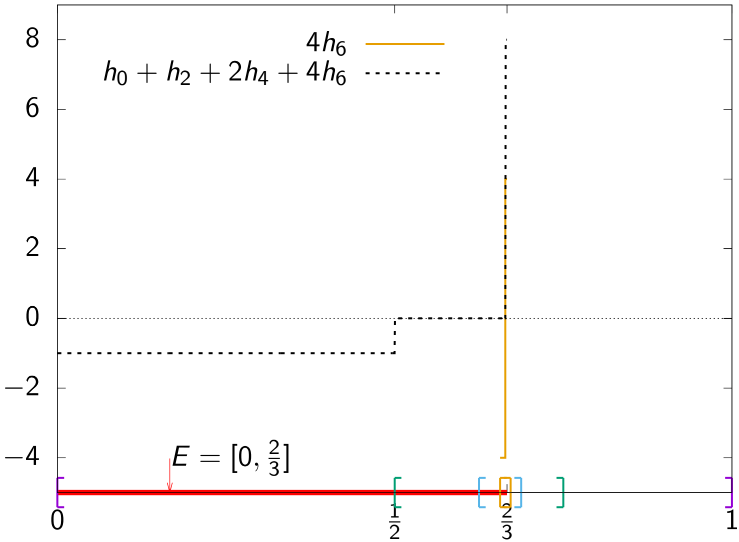

Motivation: Proof for $p = 1$

$\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| \geq 2^{-n}\}$. Goal:

$$\Big\| \sum_{I \in \mathcal{D}_n} a_I h_I \mathbb{1}_E \Big\|_{L^2}^2 \geq \sum_{I \in \mathcal{D}_n} \|a_I h_I \mathbb{1}_E\|_2^2$$

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Induction on n:

$$\|\sum_{J \in \mathcal{D}_0} a_J h_J \mathbb{1}_E\|_{L^2}^2 \geq \sum_{J \in \mathcal{D}_0} \|a_J h_J \mathbb{1}_E\|_2^2$$

$$\|\sum_{J \in \mathcal{D}_{n+1}} a_J h_J \mathbb{1}_E\|_{L^2}^2 - \|\sum_{J \in \mathcal{D}_n} a_J h_J \mathbb{1}_E\|_{L^2}^2 \geq \sum_{J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n} \|a_J h_J \mathbb{1}_E\|_2^2$$

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$\mathcal{D}_{n+1} \setminus \mathcal{D}_n$ partitions $[0,1]$. So it suffices to show for each $I \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$

$$\|\sum_{J\in\mathcal{D}_n} a_J h_J \mathbb{1}_E + a_I h_I \mathbb{1}_E\|_{L^2(I)}^2 - \|\sum_{J\in\mathcal{D}_n} a_J h_J \mathbb{1}_E\|_{L^2(I)}^2 \geq \|a_I h_I \mathbb{1}_E\|_2^2$$

Proof for $p > \frac{2}{3}$

$\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| \geq 2^{-n}\}$. Goal:

$$\| \sum_{I \in \mathcal{D}_n} a_I h_I \mathbb{1}_E \|^2_{L^2(w_n)} \geq \sum_{I \in \mathcal{D}_n} \|a_I h_I \mathbb{1}_E\|_2^2$$

Induction on n:

$$\| \sum_{J \in \mathcal{D}_0} a_J h_J \mathbb{1}_E \|^2_{L^2(w_0)} \geq \sum_{J \in \mathcal{D}_0} \|a_J h_J \mathbb{1}_E\|_2^2$$

$$\| \sum_{J \in \mathcal{D}_{n+1}} a_J h_J \mathbb{1}_E \|^2_{L^2(w_{n+1})} - \| \sum_{J \in \mathcal{D}_n} a_J h_J \mathbb{1}_E \|^2_{L^2(w_n)} \geq \sum_{J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n} \|a_J h_J \mathbb{1}_E\|_2^2$$

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$$\| \sum_{J \in \mathcal{D}_n} a_J h_J \mathbb{1}_E + a_I h_I \mathbb{1}_E \|^2_{L^2(I, w_{n+1})} - \| \sum_{J \in \mathcal{D}_n} a_J h_J \mathbb{1}_E \|^2_{L^2(I, w_n)} \geq \|a_I h_I \mathbb{1}_E\|_2^2$$

Proof for $p > \frac{2}{3}$

$\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| \geq 2^{-n}\}$. Goal:

$$\|\sum_{I\in\mathcal{D}_n} a_I h_I \mathbb{1}_E\|^2_{L^2(w_n)} \geq \sum_{I\in\mathcal{D}_n} \|a_I h_I \mathbb{1}_E\|_2^2$$

Induction on n:

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$$1 \leq w_n \leq C.$$

Proof for $p > \frac{2}{3}$

$\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| \geq 2^{-n}\}$. Goal:

$$C \| \sum_{I \in \mathcal{D}_n} a_I h_I \mathbb{1}_E \|_2^2 \geq \| \sum_{I \in \mathcal{D}_n} a_I h_I \mathbb{1}_E \|_{L^2(w_n)}^2 \geq \sum_{I \in \mathcal{D}_n} \| a_I h_I \mathbb{1}_E \|_2^2$$

Induction on n:

$$\| \sum_{J \in \mathcal{D}_0} a_J h_J \mathbb{1}_E \|_{L^2(w_0)}^2 \geq \sum_{J \in \mathcal{D}_0} \| a_J h_J \mathbb{1}_E \|_2^2$$

$$\| \sum_{J \in \mathcal{D}_{n+1}} a_J h_J \mathbb{1}_E \|_{L^2(w_{n+1})}^2 - \| \sum_{J \in \mathcal{D}_n} a_J h_J \mathbb{1}_E \|_{L^2(w_n)}^2 \geq \sum_{J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n} \| a_J h_J \mathbb{1}_E \|_2^2$$

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$$1 \leq w_n \leq C.$$

And after rescaling

$$\|\mathbb{1}_I\mathbb{1}_E + xh_I\mathbb{1}_E\|^2_{L^2(w_{n+1})} - \|\mathbb{1}_I\mathbb{1}_E\|^2_{L^2(w_n)} \geq \|xh_I\mathbb{1}_E\|^2_2$$

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$$\|\mathbb{1}_I \mathbb{1}_E + x h_I \mathbb{1}_E\|^2_{L^2(w_{n+1})} - \|\mathbb{1}_I \mathbb{1}_E\|^2_{L^2(w_n)} \geq \|x h_I \mathbb{1}_E\|_2^2$$

$$\text{on } J \in \mathcal{D}_{k+1} \setminus \mathcal{D}_k : \qquad w_k = f\left(\frac{|J \cap E|}{|J|}\right)$$

And after rescaling

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$$\text{on } J \in \mathcal{D}_{k+1} \setminus \mathcal{D}_k : \qquad w_k = f\left(\frac{|J \cap E|}{|J|}\right)$$

$$q_1 := \frac{|I_{\mathfrak{l}} \cap E|}{|I_{\mathfrak{l}}|} \qquad q_2 := \frac{|I_{\mathfrak{r}} \cap E|}{|I_{\mathfrak{r}}|} \qquad \frac{q_1 + q_2}{2} = \frac{|I \cap E|}{|I|}$$

And after rescaling

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$$|I_{\mathfrak{l}}| q_1 f(q_1)(1-x)^2 + |I_{\mathfrak{r}}| q_2 f(q_2)(1+x)^2 - |I| \frac{q_1+q_2}{2} f(\frac{q_1+q_2}{2})$$
$$\geq |I| \frac{q_1+q_2}{2} x^2$$

And after rescaling

$$\|\mathbb{1}_I \mathbb{1}_E + x h_I \mathbb{1}_E\|_{L^2(w_{n+1})}^2 - \|\mathbb{1}_I \mathbb{1}_E\|_{L^2(w_n)}^2 \geq \|x h_I \mathbb{1}_E\|_2^2$$

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$$\geq |I| \frac{q_1+q_2}{2} x^2$$

$$g(q) = q f(q)$$

And after rescaling

$$\|\mathbb{1}_I\mathbb{1}_E + xh_I\mathbb{1}_E\|^2_{L^2(w_{n+1})} - \|\mathbb{1}_I\mathbb{1}_E\|^2_{L^2(w_n)} \geq \|xh_I\mathbb{1}_E\|^2_2$$

$$\text{on } J \in \mathcal{D}_{k+1} \setminus \mathcal{D}_k : \qquad w_k = f\left(\frac{|J \cap E|}{|J|}\right)$$

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$$\frac{1}{2}g(q_1)(1-x)^2 + \frac{1}{2}g(q_2)(1+x)^2 - g\left(\frac{q_1+q_2}{2}\right) \geq x^2$$

$$g(q) = qf(q)$$

Find $g : [0,1] \to \mathbb{R}$ with:

If $q_1, q_2 \in [0,1]$ and

$$\frac{q_1 + q_2}{2} \geq p \qquad \text{or} \qquad x = 0$$

then

$$\frac{1}{2}g(q_1)(1-x)^2 + \frac{1}{2}g(q_2)(1+x)^2 - g(\frac{q_1+q_2}{2}) \geq x^2 \tag{1}$$

Find $g : [0,1] \to \mathbb{R}$ with:

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If $q \in [0,1]$ then

$$q \leq g(q) \leq Cq \tag{2}$$

Find $g : [0,1] \to \mathbb{R}$ with:

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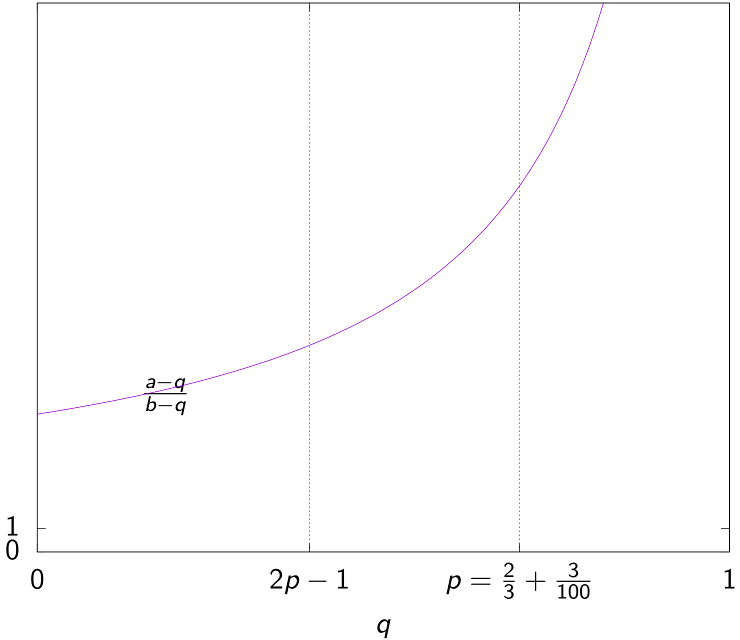
If $q \in [0,1]$ then

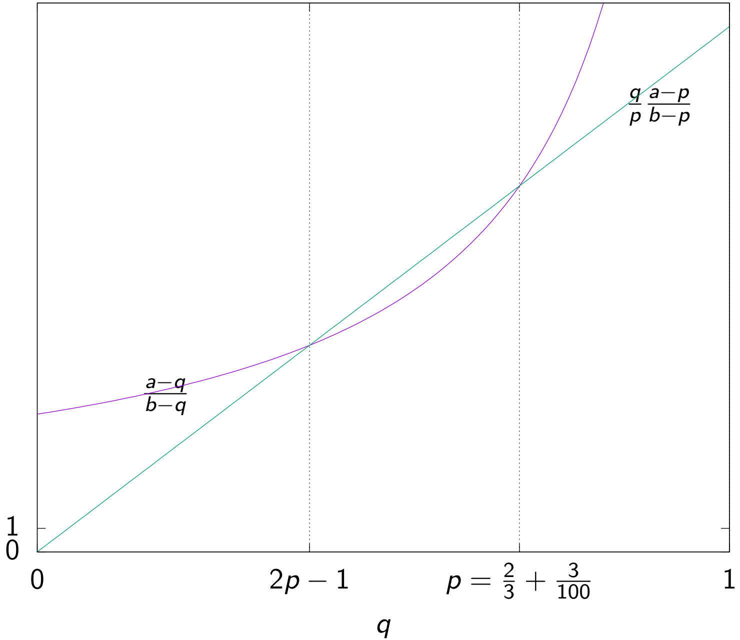
$$q \leq g(q) \leq Cq \tag{2}$$

If we set

$$g(q) := \frac{a-q}{b-q}$$

with $a \geq b > 1$ then g satisfies (**??**) for all $q_1, q_2 \in [0,1]$. But not (**??**).


$\frac{a-q}{b-q}$
1
0
0
$2p-1$
$p = \frac{2}{3} + \frac{3}{100}$
1
q


$\frac{q}{p}\frac{a-p}{b-p}$
$\frac{a-q}{b-q}$
1
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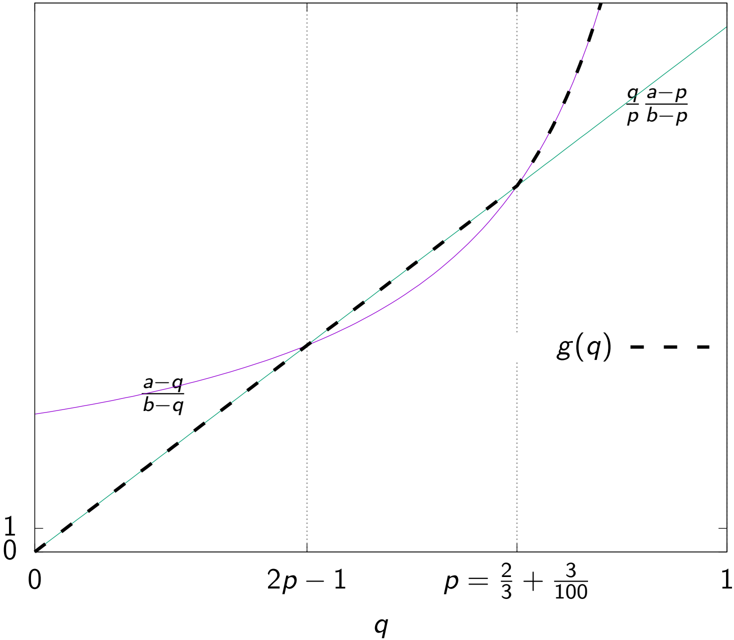
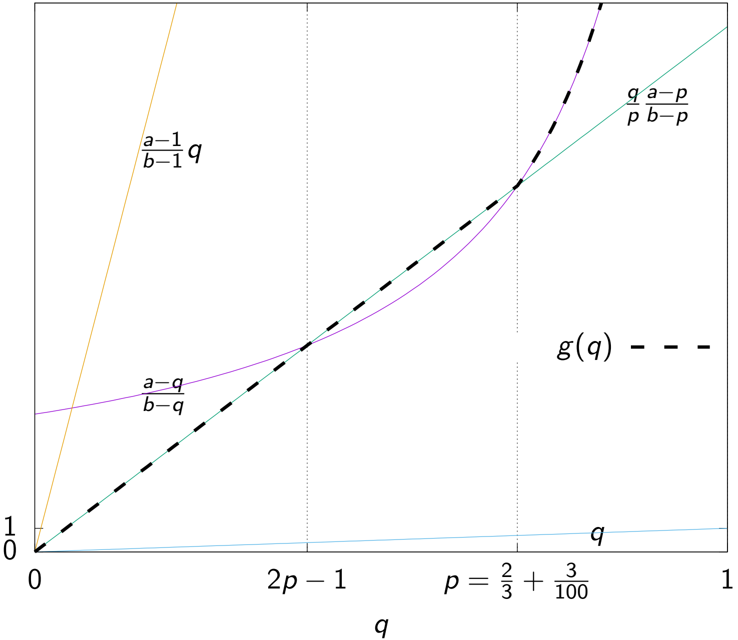

$\frac{q}{p}\frac{a-p}{b-p}$
$g(q)$
$\frac{a-q}{b-q}$
1
0
0
$2p-1$
$p = \frac{2}{3} + \frac{3}{100}$
1
q



$\frac{a-1}{b-1}q$
$\frac{q}{p}\frac{a-p}{b-p}$
$\frac{a-q}{b-q}$
$g(q)$
q
1
0
0
$2p-1$
$p=\frac{2}{3}+\frac{3}{100}$
1
q

$$
\begin{aligned}
C = g(1) &= \frac{a-1}{b-1} \\
&= \frac{8}{81}\left(p - \frac{2}{3}\right)^{-2} + \mathcal{O}\left(p - \frac{2}{3}\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
C = g(1) &= \frac{a-1}{b-1} \\
&= \frac{8}{81}\left(p - \frac{2}{3}\right)^{-2} + \mathcal{O}\left(p - \frac{2}{3}\right)^{-1} \\
C_{\text{opt}} &= \frac{1}{27}\left(p - \frac{2}{3}\right)^{-2} + \mathcal{O}\left(p - \frac{2}{3}\right)^{-1} \quad ?
\end{aligned}
$$