

# Weighted and fractional Poincaré Inequalities

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based on work with

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# Classical Poincaré

## Poincaré inequality

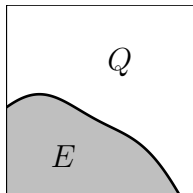
For  $1 \leq p \leq d$  and  $p \leq q \leq p^*$  we have

$$\left( \int_Q |f - f_Q|^q \right)^{\frac{1}{q}} \lesssim_d |Q|^{\frac{d}{q} - \frac{d}{p^*}} \left( \int_Q |\nabla f|^p \right)^{\frac{1}{p}}$$

with  $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$  and  $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$ . For  $p = 1$  it's equivalent to

## relative isoperimetric inequality

$$\min \left\{ \mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E) \right\}^{d-1} \lesssim_d \mathcal{H}^{d-1}(Q \cap \partial E)^d$$



Theorem (Bourgain, Brezis, and Mironescu 2002; Maz'ya and Shaposhnikova 2002; Ponce 2004; Milman 2005)

Let  $0 \leq \delta < 1$ . Then

$$\int_Q |f - f_Q| \lesssim_d (1 - \delta) l(Q)^\delta \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{d+\delta}} dx dy \lesssim l(Q) \int_Q |\nabla f|$$

$L^p$  version with  $\frac{1}{p_\delta^*} = \frac{1}{p} - \frac{\delta}{d}$ .

## With weights

For  $0 \leq \alpha \leq d$  the fractional maximal function is

$$M_\alpha \mu(x) = \sup_{r>0} r^\alpha \frac{\mu(B(x, r))}{\mathcal{L}(B(x, r))}.$$

Theorem (Franchi, Pérez, and Wheeden 2000)

Let  $1 \leq q \leq \frac{d}{d-1}$ . Then

$$\left( \int_Q |f - f_Q|^q d\mu \right)^{\frac{1}{q}} \lesssim_{d,q} \int_Q |\nabla f(x)| M_{d-q(d-1)} \mu(x)^{\frac{1}{q}} dx.$$

- Constant blows up for  $q \searrow 1$ , but is finite for  $q = 1$ .
- Generalizes Meyers and Ziemer 1977 who consider  $\mu(x) \lesssim |x|^{-\alpha}$  which implies  $M_\alpha \mu \lesssim 1$ .

## Theorem (Myrskyläinen, Pérez, and Weigt 2023)

Let  $0 \leq \delta < 1$  and  $1 \leq q \leq \frac{d}{d-\delta}$ . Then

$$\left( \int_Q |f - f_Q|^q \mu \right)^{\frac{1}{q}} \lesssim_d (1 - \delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x - y|^{n+\delta}} dy M_{d-q(d-\delta)} \mu(x)^{\frac{1}{q}} dx$$

- Hurri-Syrjänen, Martínez-Perales, Pérez, and Vähäkangas 2022: case  $q = \frac{d}{d-\delta}$ .
- Implies Franchi, Pérez, and Wheeden 2000 without blowup at  $p \rightarrow 1$ .
- $d - q(d - \delta)$  is optimal.

$p > 1$ ?

### Counterexample (Myrskyläinen, Pérez, and Weigt 2023)

The corresponding weighted Poincaré inequality does **not** hold for  $p > 1$ .

Is also counterexample against weighted fractional  $p$ -Poincaré for  $\delta$  near 1.

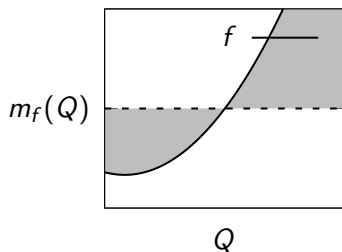
### Theorem (Hurri-Syrjänen, Martínez-Perales, Pérez, and Vähäkangas 2022)

$$\left( \int_Q |f - f_Q|^p \right)^{\frac{1}{p}} \lesssim_d (1 - \delta)^{\frac{1}{p}} \frac{|(Q)^\varepsilon|}{\varepsilon} \left( \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{d + \delta p}} dx M_{(\delta - \varepsilon)p} \mu(y) dy \right)^{\frac{1}{p}}$$

- Not optimal for  $p = 1$  by our result.
- Is there a unified result for all  $p$ ?

# Classical Poincaré by isoperimetric inequality

Consider median  $m_f(Q)$  instead of the average



$$\blacksquare = \int_Q |f - m_f(Q)| = \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\}) d\lambda + \int_{-\infty}^{m_f(Q)} \mathcal{L}(\{x \in Q : f(x) < \lambda\}) d\lambda$$

# Classical Poincaré by isoperimetric inequality

$$\begin{aligned} & \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\}) \, d\lambda \\ & \leq I(Q) \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})^{\frac{d-1}{d}} \, d\lambda \\ & \lesssim I(Q) \int_{m_f(Q)}^{\infty} \mathcal{H}^{d-1}(\partial\{x \in Q : f(x) > \lambda\}) \, d\lambda \\ & \leq I(Q) \int_Q |\nabla f| \end{aligned}$$

□

general measure: replace  $\mathcal{L} \rightarrow \mu$ , weigh  $\mathcal{H}^{d-1}$  with  $M_\alpha \mu$ .

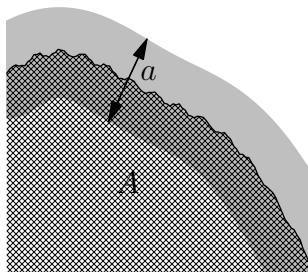


# Fractional isoperimetric inequality

Lemma (Fractional relative isoperimetric inequality)

Let  $a > 0$  and  $A \subset Q$  with  $a^d \leq \mathcal{L}(A) \leq \mathcal{L}(Q)/2$ . Then

$$\begin{aligned} a\mathcal{L}(Q \cap A)^{\frac{d-1}{d}} &\lesssim \int_Q \int_{Q \cap B(x,a) \setminus B(x,a/2)} |1_A(x) - 1_A(y)| \, dy \, dx \\ &\lesssim a\mathcal{H}^{d-1}(Q \cap \partial A) \end{aligned}$$



Thank you