Weighted and fractional Poincaré Inequalities

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based on work with

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and

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Classical Poincaré

Poincaré inequality

For $1 \leq p \leq d$ and $p \leq q \leq p^*$ we have

$$\left(\int_Q |f - f_Q|^q\right)^{\frac{1}{q}} \lesssim_d \mathrm{l}(Q)^{\frac{d}{q} - \frac{d}{p^*}} \left(\int_Q |\nabla f|^p\right)^{\frac{1}{p}}$$

with $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

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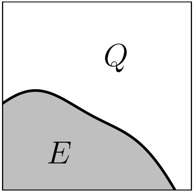
with $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$. For $p = 1$ it's equivalent to

relative isoperimetric inequality

$$\min\left\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\right\}^{d-1} \lesssim_d \mathcal{H}^{d-1}(Q \cap \partial E)^d$$



Fractional Poincaré

Theorem (Bourgain, Brezis, and Mironescu 2002; Maz'ya and Shaposhnikova 2002; Ponce 2004; Milman 2005)

Let $0 \leq \delta < 1$. *Then*

$$\int_Q |f - f_Q| \lesssim_d (1-\delta)\, \mathrm{l}(Q)^\delta \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{d+\delta}} \,\mathrm{d}x \,\mathrm{d}y \lesssim \mathrm{l}(Q) \int_Q |\nabla f|$$

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L^p version with $\frac{1}{p_\delta^*} = \frac{1}{p} - \frac{\delta}{d}$.

With weights

For $0 \leq \alpha \leq d$ the fractional maximal function is

$$\mathrm{M}_\alpha \mu(x) = \sup_{r>0} r^\alpha \frac{\mu(B(x,r))}{\mathcal{L}(B(x,r))}.$$

Theorem (Franchi, Pérez, and Wheeden 2000)

Let $1 \leq q \leq \frac{d}{d-1}$. Then

$$\left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \lesssim_{d,q} \int_Q |\nabla f(x)| \mathrm{M}_{d-q(d-1)}\mu(x)^{\frac{1}{q}} \,\mathrm{d}x.$$

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- Constant blows up for $q \searrow 1$, but is finite for $q = 1$.

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- Constant blows up for $q \searrow 1$, but is finite for $q = 1$.
- Generalizes Meyers and Ziemer 1977 who consider $\mu(x) \lesssim |x|^{-\alpha}$ which implies $\mathrm{M}_\alpha \mu \lesssim 1$.

Fractional with weights

Theorem (Myyryläinen, Pérez, and Weigt 2023)

Let $0 \leq \delta < 1$ *and* $1 \leq q \leq \frac{d}{d-\delta}$. *Then*

$$\Big(\int_Q |f - f_Q|^q \mu \Big)^{\frac{1}{q}} \lesssim_d$$
$$(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y \, \mathrm{M}_{d-q(d-\delta)} \mu(x)^{\frac{1}{q}} \,\mathrm{d}x$$

- Hurri-Syrjänen, Martínez-Perales, Pérez, and Vähäkangas 2022: case $q = \frac{d}{d-\delta}$.

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- Implies Franchi, Pérez, and Wheeden 2000 without blowup at $p \to 1$.
- $d - q(d-\delta)$ is optimal.

$p > 1$?

Counterexample (Myyryläinen, Pérez, and Weigt 2023)

The corresponding weighted Poincaré inequality does **not** hold for $p > 1$.

Is also counterexample against weighted fractional p-Poincaré for δ near 1.

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- Not optimal for $p = 1$ by our result.

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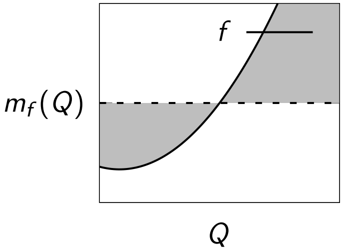
- Not optimal for $p = 1$ by our result.
- Is there a unified result for all p?

Classical Poincaré by isoperimetric inequality

Consider median $m_f(Q)$ instead of the average



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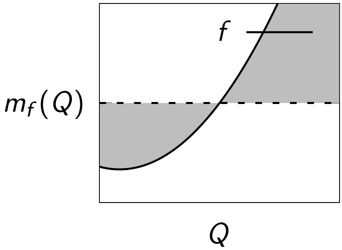

$$\blacksquare = \int_Q |f - m_f(Q)| = \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})\,\mathrm{d}\lambda + \int_{-\infty}^{m_f(Q)} \mathcal{L}(\{x \in Q : f(x) < \lambda\})\,\mathrm{d}\lambda$$

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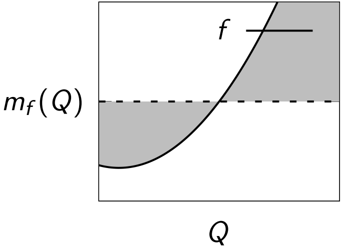

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$$\int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})\,\mathrm{d}\lambda$$

$$\leq \mathsf{I}(Q) \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})^{\frac{d-1}{d}}\,\mathrm{d}\lambda$$

Classical Poincaré by isoperimetric inequality

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$\square$

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□

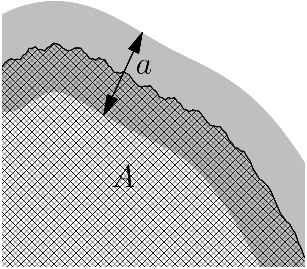
general measure: replace $\mathcal{L} \to \mu$, weigh $\mathcal{H}^{d-1}$ with $\mathrm{M}_\alpha \mu$.

Fractional isoperimetric inequality

Lemma (Fractional relative isoperimetric inequality)

Let $a > 0$ and $A \subset Q$ with $a^d \leq \mathcal{L}(A) \leq \mathcal{L}(Q)/2$. Then

$$a\mathcal{L}(Q \cap A)^{\frac{d-1}{d}} \lesssim \int_Q \fint_{Q \cap B(x,a)} |1_A(x) - 1_A(y)| \,\mathrm{d}y \,\mathrm{d}x$$



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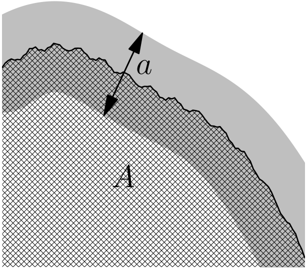

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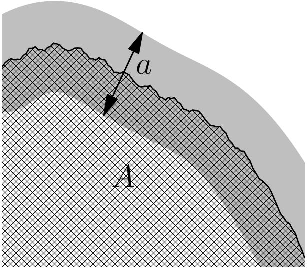

Thank you