9. Lower bound for vertex Δ-coloring in the LOCAL model

By the result of Linial [Lin92], we know that there is a deterministic local algorithm $\mathcal{A}$ of complexity $O(\log^* n)$ that solves vertex $(\Delta+1)$-coloring problem on finite graphs of degree bounded by Δ. By Brooks' Theorem, there is a vertex Δ-coloring of a connected graph G of degree bounded by Δ if and only if G is not a complete graph or a cycle of odd length. It is a natural question if there is an efficient algorithm that finds Δ-coloring on e.g. *finite trees.* We use the ideas of Marks [Mar16] to show that there is no such deterministic algorithm of complexity $O(\log^* n)$.

Theorem 0.1 ([BFH$^+$16, CKP19, Mar16]). *The deterministic complexity of the vertex Δ-coloring problem on finite trees of degree bounded by Δ is $\Omega(\log n)$.*

If it were not the case, then by the speed up theorem there is a deterministic local algorithm $\mathcal{A}$ that has complexity $O(\log^* n)$. We show that any such algorithm has to fail on a finite tree that is additionally endowed with a proper Δ-edge coloring. So for a contradiction fix such an algorithm $\mathcal{A}$.

Let T_Δ be a rooted infinite Δ-regular tree endowed with a proper Δ-edge coloring. Write $T_{\Delta,n}$ for the $O(\log^* n)$ neighborhood of the root. Inspired by Marks, we define the following games $\mathbb{G}_n(\ell,\alpha)$, where $n \in \mathbb{N}$, $\ell \in n$ and $\alpha \in \Delta$. In the initial step, the root is labeled by ℓ, then Alice and Bob alternate and label some vertices with unique identifiers from n. Namely, in the k-th step, Alice first labels all vertices of distance k from the root in the direction of α edge, and then Bob labels the remaing vertices of distance k from the root. The game proceeds for $O(\log^* n)$ rounds. After that we say that Alice wins if $\mathcal{A}(T_{\Delta,n}) \neq \alpha$, i.e., if the color at the root that is produced by $\mathcal{A}$ when applied to $T_{\Delta,n}$ with the labeling produced by the run of the game is not α.

Following Marks' argument, we would like to deduce that for every $\ell \in \mathbb{N}$ there is α so that Bob has a winning strategy in the game $\mathbb{G}_n(\ell,\alpha)$. Unfortunately, as the original argument combines several strategies it is easy to see that this might not produce unique labeling. We circumvent these difficulties by requiring additional restrictions on allowed moves.

ID graphs Let $(H_n)_n$ be a sequence of graphs with edge Δ-labeling that satisfy the following:

1. the girth of each H_n is at least $O(\log^* n)$,
2. the size of each H_n is at most n,
3. each vertex is adjacent to at least one edge of each label from Δ,
4. for each $i \in \Delta$, the graph H_n^i (the restriction of H_n to i-edges) has the independence ratio at most $1/\Delta$.

We call such a sequence an *ID graph.* We require that the constant in 1) is slightly bigger than the one in the algorithm $\mathcal{A}$. Also we assume that the vertices of H_n are labeled uniquely by elements from n.

Claim 0.2. *For every Δ, there is an ID graph $(H_n)_n$.*

Proof. Use the configuration model for producing ranomd graphs, i.e., sample enough random perfect matchings. □

The argument We modify the games $\mathbb{G}_n(\ell, \alpha)$ in an obvious way, namely the produced labeling has to be a homomorphism from $T_{\Delta,n}$ to H_n that sends root to ℓ and preserves edge labels.

Claim 0.3. *For every $\ell \in n$ there is α so that* Bob *has a winning strategy in $\mathbb{G}_n(\ell, \alpha)$.*

Proof. Suppose not. Then as the games are finite Alice has a winning strategies all the games $\mathbb{G}(\ell, \beta)$, where $\beta \in \Delta$. Playing strategies of Alice against each other produces a labeling that satisfies $\mathcal{A}(T_{\Delta,n}) \neq \beta$ for every $\beta \in \Delta$, a contradiction. □

Write $c : H_n \to \Delta$ for the map $\ell \mapsto \alpha$, where Bob has a winning strategy in $\mathbb{G}_n(\ell, \alpha)$. Note that this is well defined. By the pigeonhole principle and property 4) of the ID graph we have that there is an $\alpha \in \Delta$ and an α-edge $\{\ell_0, \ell_1\}$ so that Bob wins both $\mathbb{G}(\ell_i, \alpha)$, where $i \in \{0, 1\}$. Consider now the graph $\widetilde{T}_{\Delta,n}$ that is "rooted" at an α-edge and contains both graphs $T_{\Delta,n}$ rooted at the endpoints of this edge. Label these endpoints by ℓ_0 and ℓ_1. Playing the two strategies of Bob against each other one rooted at ℓ_0 and the other one at ℓ_1 produces some labeling of $\widetilde{T}_{\Delta,n}$. It is easy to see that both endpoints of the "rooted" edge are assigned color α by $\mathcal{A}$, a contradiction.

References

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