8. Borel colorings

A set X together with a Borel σ-algebra $\mathcal{B}$ is called a *standard Borel space* if there is a completely metrizable and separable topology τ on X so that $\mathcal{B}$ coincide with the σ-algebra of τ-Borel sets, i.e., the minimal σ-algebra that contains all τ-open sets. By the Borel isomorphism theorem, we have that two standard Borel spaces $(X.\mathcal{B})$ and $(Y, \mathcal{C})$ are Borel isomorphic, i.e., there is a Borel measurable bijection $f : X \to Y$, if and only if $|X| = |Y|$. Moreover, we have that $|X|$ is finite, countable or of the size of the continuum. That is, up to isomorphism, $(X, \mathcal{B})$ is either $N \in \mathbb{N} \cup \{\mathbb{N}\}$ endowed with the σ-algebra of all subsets, or $[0, 1]$ with the Borel σ-algebra generated by open intervals.

A *Borel graph* $\mathcal{G}$ on a standard Borel space X is a symmetric, irreflexive and Borel subset of $X \times X$. We say that the degree of $\mathcal{G}$ is bounded by $\Delta < \infty$ if $|\{y \in X : (x, y) \in \mathcal{G}\}| \leq \Delta$ for every $x \in X$. If every degree is finite, we say that $\mathcal{G}$ is *locally finite*, and if every degree is at most countable, se say that $\mathcal{G}$ is *locally countable*. Note that in all these cases, we have that $[x]_{\mathcal{G}}$, the connectivity component of x in $\mathcal{G}$, is at most countable. In another words, $\mathcal{G}$ is a disjoint union of at most countable graphs, if X is uncountable, then there are uncountably many such countable graphs.

A typical examples are (Borel) Schreier graphs of actions of countable groups. Let (Γ, S) be a finitely generated group and $\Gamma \curvearrowright X$ be a Borel action, that is, $x \mapsto \gamma \cdot x$ is a Borel map for every $\gamma \in \Gamma$. Define the Schreier graph of the action as follows $(x, y) \in \mathrm{Sch}(X; \Gamma, S)$ if and only if there is $s \in S \cup S^{-1}$ so that $s \cdot x = y$. It is easy to see that $\mathrm{Sch}(X; \Gamma, S)$ is a Borel graph if degree bounded by $2|S|$. We say that the action $\Gamma \curvearrowright X$ is *free* if $\gamma \cdot x = x$ implies $\gamma = 1_\Gamma$ for every $\gamma \in \Gamma$ and $x \in X$. In that case, $[x]_{\mathrm{Sch}(X;\Gamma,S)}$ is isomorphic to $\mathrm{Cay}(\Gamma, S)$ for every $x \in X$.

Given an LCL $\Pi = (b, t, \mathcal{P})$ and a Borel graph $\mathcal{G}$, we investigate when there is a Borel Π-coloring of $\mathcal{G}$. That is, a Borel function $f : X \to b$ that satisfies constraints $\mathcal{P}$ around each $x \in X$, here we view the finite set b as a standard Borel space. Note that when $\mathcal{G}$ is a Borel locally finite graph, then $x \mapsto \mathcal{B}_{\mathcal{G}}(x, , r)$ is a Borel map for every $r \in \mathbb{N}$. For a fixed class of Borel graphs, e.g., uniformly bounded degree by $\Delta < \infty$, acyclic, or Schreier graphs induced by free actions of (Γ, S), we say that $\Pi \in \mathsf{BOREL}$ if every $\mathcal{G}$ in the class admits a Borel Π-coloring.

As usual we start with the following.

Theorem 0.1 (Kechris–Solecki–Todorcevic)**.** *Let Π be the proper vertex $(\Delta + 1)$-coloring problem. Then $\Pi \in \mathsf{BOREL}$ for the class of all Borel graphs of degree bounded by Δ.*

Proof. Let $\mathcal{G}$ be such Borel graph on a standard Borel space X. Let $(A_i)_{i \in \mathbb{N}}$ be a basis for some Polish topology τ that generates the σ-algebra. It is easy to see that the map $x \mapsto c(x) \in \mathbb{N}$, where $c(x) \in \mathbb{N}$ is minimal such that $x \in A_{c(x)}$ but $N_{\mathcal{G}}(x) \cap A_{c(x)} = \emptyset$, is Borel. Moreover, if $(x, y) \in \mathcal{G}$ implies $c(x) \neq c(y)$, i.e., c is a proper coloring. The same greedy construction as always decreases the number of colors to $\Delta + 1$. $\square$

In the case of Schreier graphs we have the following result.

Theorem 0.2 (Seward–Tucker-Drob)**.** *Let (Γ, S) be a finitely generated group and $\Gamma \curvearrowright X$ be a free Borel action. Then there is a Borel Γ-equivariant map*

$$\varphi : X \to \mathrm{Free}(\{0, 1\}^\Gamma).$$

In particular, we have $\Pi \in \mathsf{BOREL}$ if and only if there is a Borel Γ-equivariant map from $\mathrm{Free}(\{0, 1\}^\Gamma)$ to X_Π, the space of all Π-colorings.

Proof of the additional part. It is enough to show that Borel equivariant maps to X_Π are in one-to-one correspondence with Borel Π-colorings of the Schreier graph on $\mathrm{Free}(\{0,1\}^\Gamma)$.

Namely, if $F : \mathrm{Free}(\{0,1\}^\Gamma) \to b$ is a Borel Π-coloring, then

$$\varphi(x)(g) = F(g^{-1} \cdot x)$$

is a Π-coloring of $\mathrm{Cay}(\Gamma, S)$. Similarly, if $\varphi : \mathrm{Free}(\{0,1\}^\Gamma) \to X_\Pi$ is an equivariant Borel map, then $F(x) = \varphi(x)(1_\Gamma)$ is a Borel Π-coloring of the Schreier graph. □

Similar result holds for continuous colorings by [Ber21].

References

[Ber21] Anton Bernshteyn. Probabilistic constructions in continuous combinatorics and a bridge to distributed algorithms. 2021.