7) Continuous colorings

We move slowly towards *descriptive combinatorics* and the complexity classes from that area. Perhaps the easiest way to connect LOCAL model and descriptive combinatoircs is through the complexity class $\mathsf{CONTINOUS}$.

Definition 0.1. *Let* (Γ, S) *be a finitely generated group and* $\{0,1\}^\Gamma$ *be the space of all* $\{0,1\}$*-labelings of* $\mathrm{Cay}(\Gamma, S)$. *Write* $\mathrm{Free}(\{0,1\}^\Gamma)$ *for the free sub-shift of* $\{0,1\}^\Gamma$*, that is,*

$$\mathrm{Free}(\{0,1\}^\Gamma) = \{\alpha \in \{0,1\}^\Gamma : \forall 1_\Gamma \neq g \in \Gamma \; g \cdot \alpha \neq \alpha\}.$$

It is easy to see that $\mathrm{Free}(\{0,1\}^\Gamma)$ *is* Γ*-invariant.*

Remark 0.2. *We remark that passing to* $\mathrm{Free}(\{0,1\}^\Gamma)$ *is very natural, as e.g. in the iid case we have that* $\alpha \in [0,1]^\Gamma$ *has no symmetry with probability* 1.

Note that the space $\{0,1\}$ endowed with the discrete topology is compact and so is the space $\{0,1\}^\Gamma$ endowed with the product topology. The space $\mathrm{Free}(\{0,1\}^\Gamma)$ is not compact but the inhereted topology is separable, completely metrizable and zero-dimensional. Recall that *zero-dimensional* means that there are lot of sets that are closed and open = clopen. This is necessary as we would like to construct continuous map to a discrete space b, and in that case preimage of each element from b is a clopen set.

Similarly, when we endow b^Γ with the product topology, then it is easy to see that X_Π is a compact subset.

Definition 0.3. *We say that an LCL* Π *on* $\mathrm{Cay}(\Gamma, S)$ *is in the class* $\mathsf{CONTINOUS}$ *if there is a* continuous *and equivariant map* $\varphi : \mathrm{Free}(\{0,1\}^\Gamma) \to b^\Gamma$ *such that the image is contained in* X_Π.

Intuitively, continuous maps are quantitative analogous of finitary fiid. Namely, a continuous algorithm/map is given by a pair $\mathcal{A} = (\mathcal{C}, f)$, where $\mathcal{C}$ is a family of finite rooted neighborhoods with vertex labeling from $\{0,1\}$ and $f : \mathcal{C} \to b$, where b is the corresponding alphabet. A continuous algorithm/map $\mathcal{A} = (\mathcal{C}, f)$ is applied as follows: a given vertex $v \in \mathrm{Cay}(\Gamma, S)$ explores its labeled neighborhood until it encounters element from $\mathcal{C}$ and then outputs the corresponding letter from b. We require that this is well-defined on the whole $\mathrm{Free}(\{0,1\}^\Gamma)$.

Given a finite rooted neighborhood A with vertex labeling from $\{0,1\}$, $v \in \mathrm{Cay}(\Gamma, S)$ and $x \in \mathrm{Free}(\{0,1\}^\Gamma)$, we write $(v, x) \in A$ to denote that the A pattern appears around v in x.

Theorem 0.4 (Kechris-Solecki-Todorcevic; Bernshteyn)**.** *Let* (Γ, S) *be a finitely generated group and* Π *be the proper vertex* $(\Delta+1)$*-coloring problem, where* Δ *is the maximum degree in* $\mathrm{Cay}(\Gamma, S)$. *Then* $\Pi \in \mathsf{CONTINOUS}$.

Consequently, $\mathsf{LOCAL}(O(\log^* 1/\varepsilon)) \subseteq \mathsf{CONTINOUS}$*, or* $\mathsf{LOCAL}(O(\log^* n)) \subseteq \mathsf{CONTINOUS}$ *for reasonable approximation.*

Proof. Let $(A_i)_{i \in \mathbb{N}}$ be the enumeration of all finite rooted neighborhoods with vertex labeling from $\{0,1\}$. Given $v \in \mathrm{Cay}(\Gamma, S)$ and $x \in \mathrm{Free}(\{0,1\}^\Gamma)$, we let $\varphi(x)(v)$ be the minimal index $i \in \mathbb{N}$ so that $(v, x) \in A_i$ but $(w, x) \notin A_i$ for every neighbor w of v. It is easy to see that this is well defined in $\mathrm{Free}(\{0,1\}^\Gamma)$ and that it defines a proper vertex $\mathbb{N}$-coloring.

To decrease the number of colors to $(\Delta + 1)$, we apply once again the greedy algorithm inductively. Namely, if $v \in \mathrm{Cay}(\Gamma, S)$ and $x \in \mathrm{Free}(\{0,1\}^\Gamma)$ are such that $(x, v) \in A_i$ and all vertices labeled with colors $j < i$ already decreased their color, then (v, x) picks the minimal available color not used by any of its neighbors from the set $\{1, \dots, \Delta + 1\}$. $\square$

Bernshteyn [Ber21] showed that the reverse inclusion holds as well (for a suitable finite approximations of $\mathrm{Cay}(\Gamma, S)$). His argument is rather complicated and so we sketch the proof for $\mathbb{Z}^d$, in different language this is basically the "twelve tile" Theorem of Gao et al [GJKS18]. The main ingredient is the following concept: Let $\Lambda \in \{0,1\}^{\mathbb{Z}^d}$ be a labeling of $\mathbb{Z}^d$. We say that Λ is *hyperaperiodic* if for every $g \in \mathbb{Z}^d$ there is a finite set $S \subseteq \mathbb{Z}^d$ such that for every $t \in \mathbb{Z}^d$ we have

$$\Lambda(t+S) \neq \Lambda(g+t+S)$$

This notation means that the Λ-labeling of S differs from the Λ-labeling of S shifted by g everywhere. In another words, given some translation $g \in \mathbb{Z}^d$, there is a uniform witness S for the fact that Λ is nowhere g-periodic.

It follows from a standard compactness argument that any continuous algorithm $\mathcal{A}$ used on a hyperaperiodic labeling finishes after number of steps that is uniformly bounded by some constant that depends only on $\mathcal{A}$. This is because the closure of the translations of Λ, $\overline{\mathbb{Z}^d \cdot \Lambda}$, in $\{0,1\}^{\mathbb{Z}^d}$ is actually a subset of $\mathrm{Free}(\{0,1\}^{\mathbb{Z}^d})$. That is to say, the closure is a free compact subshift. It is a deep result of Gao et al [GJS09] that hyperaperiodic elements do exist for every countable group Γ.

Proposition 0.5 (Lemma 2.8 in [GJKS18])**.** *There are hyperaperiodic labelings of* $\mathbb{Z}^d$.

Proof Sketch. Let $\Lambda_0 \in \{0,1\}^{\Gamma_0}$ and $\Lambda_1 \in \{0,1\}^{\Gamma_1}$ be hyperaperiodic elements, then $\Lambda_0 \oplus \Lambda_1$ defined as $\Lambda_0 \oplus \Lambda_1(g,h) = \Lambda_0(g) + \Lambda_1(h) \mod 2$ is hyperaperidic element in $\{0,1\}^{\Lambda_0 \times \Lambda_1}$, see [GJKS18, Lemma 2.7] for proof. Therefore it is enough to find a hyperaperiodic element on $\mathbb{Z}$.

Define $\Lambda(a)$ to be the number of 1s modulo 2 in the base 2 expansion of a. □

Now we have all the ingredients to show the following.

Theorem 0.6 ([GJKS18])**.** *Let $d > 1$ and Π be an LCL on d-dimensional grids. Then we have that* $\Pi \in \mathsf{LOCAL}(O(\log^* n))$ *if and only if* $\Pi \in \mathsf{CONTINOUS}$.

Also $\Pi \in \mathsf{LOCAL}(O(1))$ *if and only if there is a uniformly continuous equivariant map* $\varphi : \mathrm{Free}(\{0,1\}^\Gamma) \to b^\Gamma$ *such that the image is contained in* X_Π.

Proof Sketch. One direction in both cases follows from the classification of such LCLs in the LOCAL model, together with Theorem 0.4. Also the uniform continuity guarantees that the solution can be extended to the whole set $\{0,1\}^\Gamma$. In particular, the value $\varphi(\mathbf{0})$ of the constant 0 labeling needs to be constant.

Let $\Pi \in \mathsf{CONTINOUS}$, $\mathcal{A}$ be the corresponding continuous algorithm and Λ by an hyperaperiodic element. For each $r \in \mathbb{N}$ consider a tiling of big enough tori with rectangles of side length in $\{r, r+1\}$. Label each tile with a pattern of $\{0,1\}$ that appears somewhere in Λ and consider the family of all local configurations of these labeled tiles. Embed this collection sparsely into Λ to create a different labeling Λ'. It can be verified that Λ' is hyperaperiodic. By the uniform continuity on Λ', there is a $w \in \mathbb{N}$ so that $\mathcal{A}$ has uniform locality w when applied to Λ'.

Pick $r \gg w$. The desired local algorithm tiles large enough tori with rectangles of side length in $\{r, r+1\}$ and assigns the chosen $\{0,1\}$-labeling on each of them. This can be done in $O(\log^* n)$-many rounds. Then apply $\mathcal{A}$ with constant locality w. It is easy to see that this produces a valid Π-coloring as all the local configurations appear in Λ'. □

References

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