5. Random processes

So far we focused on local algorithms in the context of finite graphs (deterministic or randomized local algorithms) and a certain interplay between finite and infinite graphs (uniform local algorithm). Now we start to move our attention towards infinite graphs, or in particular towards the situation when we have one fixed infinite (vertex transitive) graph. As mentioned before, it is not clear what should be the right generalization.

First we need some definitions. Fix a countable infinite graph G of degree bounded by $\Delta < \infty$, additional decorations are allowed and let $\Gamma \leq \mathrm{Aut}(G)$ be a closed subgroup (pointwise convergence). We work exclusively in the situation when Γ is countable. Think of oriented grid $\mathbb{Z}^d$ and $\Gamma = \mathrm{Aut}(\mathbb{Z}^d) = \mathbb{Z}^d$. Let A be a set and A^G be the set of all labelings of vertices of G by elements of A. There is a natural action $\Gamma \curvearrowright A^G$ defined

$$(\alpha \cdot x)(v) = x(\alpha^{-1} \cdot v)$$

for every $\alpha \in \Gamma$ and $x \in A^G$.

Definition 0.1. *Let A and B be sets. We say that a map $f : A^G \to B^G$ is Γ-equivariant if*

$$\alpha \cdot f(x) = f(\alpha \cdot x)$$

for every $x \in A^G$ and $\alpha \in \Gamma$.

Subshifts of finite type Subshifts of finite type are just LCLs in disguise. Let $\Pi = (t, b, \mathcal{P})$ be a triplet, where b is a finite set, $t \in \mathbb{N}$ and $\mathcal{P}$ is a subset of isomorphism types of t-neighborhoods of vertices from G labeled with elements of b, i.e., allowed b-configurations. Then the *subshift of finite type* is the set

$$X_\Pi = \{x \in b^G : \forall v \in V(G)\ x \upharpoonright \mathcal{B}_G(v, t) \in \mathcal{P}\},$$

that is, labelings that satisfy the constraints from $\mathcal{P}$ at every vertex, in another words, Π-colorings, together with the Γ-action. Note that X_Π is $\mathrm{Aut}(G)$-invariant, i.e., $x \in X_\Pi$ if and only if $\alpha \cdot x \in X_\Pi$ for every $\alpha \in \mathrm{Aut}(G)$. As Π posses all the information aboutX_Π we might call it as well a *subshift of finite type.*

Measurable sets Suppose that the set A is endowed with a σ-algebra of subsets that makes it standard Borel or standard probability space. In the case that A is uncountable, it is enough to think to assume that A is isomorphic to $[0, 1]$ endowed with the Borel σ-algebra that comes from the standard metric. In the case that A is at most countable, the σ-algebra that we consider is the algebra of all subsets.

The space A^G is naturally equipped with the product σ-algebra. Note that the action $\Gamma \curvearrowright A^G$ is measurable, that is, the map

$$x \mapsto \alpha \cdot x$$

is a measurable map for each $\alpha \in \Gamma$.

Random Π-coloring Let $\Pi = (b, t, \mathcal{P})$ be LCL (or subshift of finite type). Then a *random Π-coloring* is a distribution μ on b^G such that

- $\mu(X_\Pi) = 1$, i.e., it is concentrated on Π-colorings,
- μ is Γ-invariant, i.e., $\mu(S) = \mu(\alpha \cdot S)$, where $\alpha \cdot S = \{\alpha \cdot x : x \in S\}$.

Factors of IID Our task is to produce a random Π-coloring as a factor of iid labels. Here, the IID labeling on G is the product measure $\lambda^{\oplus\infty}$ on $[0,1]^{\mathbf{G}}$, where λ is the Lebesgue measure. We say that LCL $\Pi = (b, t, \mathcal{P})$ is in the class fiid if there is a measurable Γ-equivariant map $f : [0,1]^G \to b^G$ so that $\mu = f_*\lambda^{\oplus\infty}$, the push-forward of $\lambda^{\oplus\infty}$, is a random Π-coloring. In that case, we say that f solves Π. In general, a map as above is called a Γ-factor of iid if it is measurable and Γ-equivariant. If $\Gamma = \mathrm{Aut}(G)$, then we omit it and write simply factor of iid.

Definition 0.2 (Finitary fiid)**.** *A factor of iid f is called* finitary *if evaluation at every vertex depends with probability 1 on some finite neighborhood of it. That is, for each $v \in G$ there is a random variable $\mathcal{R}_f(v)$ that takes values in $\mathbb{N}$ such that*

$$x \upharpoonright \mathcal{B}_G(x, \mathcal{R}_f(v)(x)) = y \upharpoonright \mathcal{B}_G(x, \mathcal{R}_f(v)(x))$$

implies that $f(x)(v) = f(y)(v)$ for every $x, y \in [0,1]^G$.

We say that Π is in the class ffiid *if there is a finitary fiid f that solves Π.*

Distinguishing fiid and ffiid is the biggest open problem, not known to be different in any case for LCLs without inputs. They do differ on lines when we consider LCLs with inputs. We believe that they are the same on grids (maybe amenable graphs) and different on trees, e.g., what about perfect matching.

Cayley graphs Let (Γ, S) be a finitely generated group, i.e., Γ is a group and $1_\Gamma \notin S = S^{-1}$ is a generating set. We write $\mathrm{Cay}(\Gamma, S)$ for the (right) Cayley graph of (Γ, S), that is, $\alpha, \beta \in \Gamma$ form an σ-edge (oriented) if there is $\sigma \in S$ such that $\alpha \cdot \sigma = \beta$. Then $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S)) = \Gamma$. This is because an element of $\mathrm{Aut}(\mathrm{Cay}(\Gamma, S))$ is fully determined once we know where 1_Γ goes.

Proposition 0.3. *Let (Γ, S) be a finitely generated group, b a finite set and set $G = \mathrm{Cay}(\Gamma, S)$. There is a one-to-one correspondence between finitary fiid and uniform local algorithms on $\mathbf{G} = \{G\}$ with range b^G. Moreover, given ffiid f or uniform local algorithm $\mathcal{A}$ the coding radius is the same at each vertex, i.e., $\mathcal{R}_f(v)$, or $\mathcal{R}_{\mathcal{A}}(v)$, is independent of $v \in G$.*

Proof. Let $\mathcal{A}$ be a uniform algorithm that has the finite set b as its output alphabet. Given $x \in [0,1]^G$, we run $\mathcal{A}$ at each vertex v simultaneously. This produces with probability 1 an element $f(x) = \mathcal{A}(G)(x) \in b^G$. As $\mathcal{A}$ considers isomorphism classes of rooted neighborhoods as an input, we see that

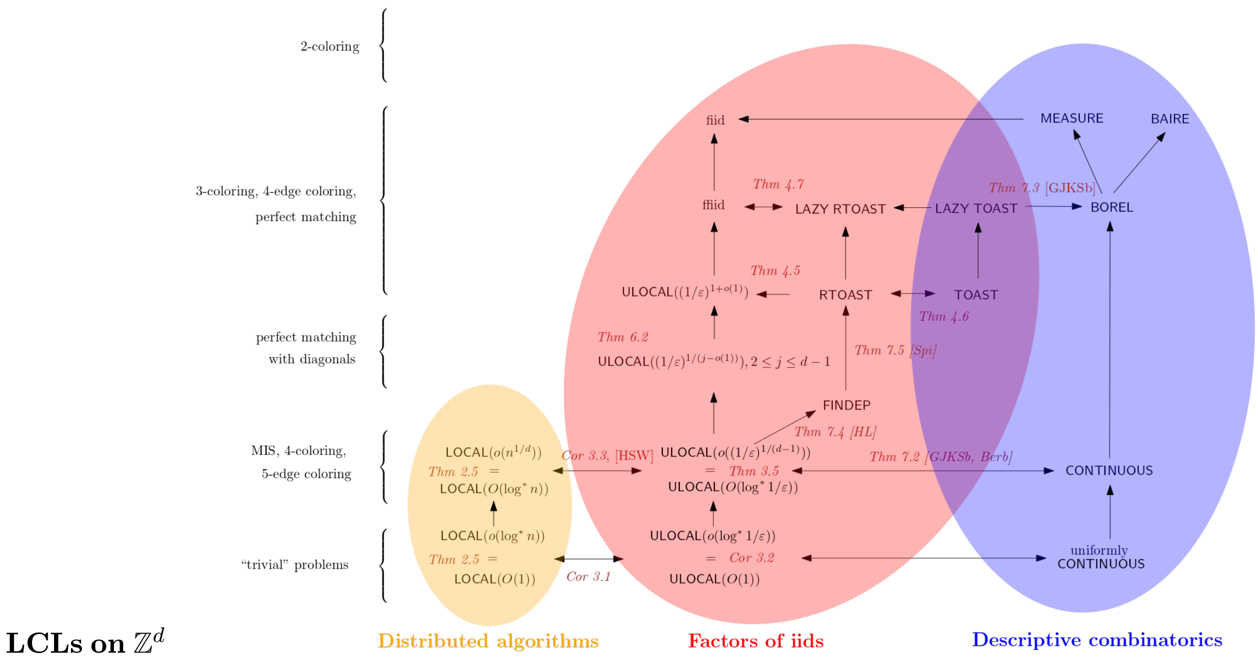
$$f : [0,1]^G \to b^G$$

is Γ-invariant and finitary.

Let $f : [0,1]^G \to b^G$ be a ffiid. Collect all the finite neighborhoods that decide the value at 1_Γ together with the output assignemnet. That is, if $x \upharpoonright \mathcal{B}_G(1_\Gamma, t)$ is as in the definition of ffiid, then we add it in the domain of $\mathcal{A}$. By the Γ-invariance and the fact that $\Gamma = \mathrm{Aut}(G)$, this defines the same function regardless of the starting vertex. The fact that $\mathcal{A}$ produces an output with probability one follows from the fact that f is such. $\square$

This allows to measure complexity of an ffiids or LCLs as in the ULOCAL regime. Moreover, we might consider the following, coarser, quantity to measure complexity.

Definition 0.4 (Moments)**.** *Let f be ffiid, then we write $\mathbf{M}(f)$ to denote the supremum of finite moments of the coding radius $\mathcal{R}_f$. Similarly, if Π is an LCL, then we denote $\mathbf{M}(\Pi)$ to be the supremum of all $\mathbf{M}(f)$, where f solves Π.*



3-coloring of grids Let $d > 1$ and write $\mathbb{Z}^d$ for the d-dimensional oriented infinite grid. It was shown, independently, that that the complexity of the 4-coloring problem is $O(\log^* n)$ in the LOCAL model [BHK$^+$17], and, in our language, $O(\log^* 1/\varepsilon)$ in the uniform local model [HSW17]. That is, [HSW17] showed that there is a ffiid that produces a 4-coloring and has such a tail decay of the coding radius.

Concerning the 3-coloring problem Π, [BHK$^+$17] showed that in the local model the complexity is $\Omega(\sqrt[d]{n})$ and [HSW17] that $0 < \mathbf{M}(\Pi) \le 2$. As an application of the approach that combines both techniques we show the following.

Theorem 0.5. *Let Π be the proper vertex 3-coloring problem. Then we have*

$$\Pi \in \mathsf{ULOCAL}((1/\varepsilon)^{1+o(1)})$$

on the d-dimensional infinite grid (for any dimension $d > 1$). In particular, $\mathbf{M}(\Pi) = 1$.

a) To produce such a coloring we borrow an approach from descriptive combinatorics called TOAST construction. Let $\mathcal{T}$ be a hierarchy with boundaries far apart (called *toast*) and consider the algorithm that colors recursively as follows. It tries to do 2-coloring but if there is a parity collision with previous decision, then it uses the third color to change the parity.

b) Standard way to produce toast via Voronoi cells can be done in a finitary fashion with coding radius exactly $(1/\varepsilon)^{1+o(1)}$. Note that in the construction we need to use MIS with larger and larger parameter.

These two steps combined shows upper bound on the complexity of Π.

c) To get the lower bound we use a technique from LOCAL model. It is enough to work on 2-dimensional grids. We show that any uniform algorithm $\mathcal{A}$ solving 3-coloring that satisfies

$$\mathrm{P}(\mathcal{R}_{\mathcal{A}} > C1/\varepsilon) \le \varepsilon,$$

for an absolute constant $C > 0$, can be turned into a deterministic local algorithm that solves 3-coloring on finite tori in $O(\log^* n)$ many rounds. That is not possible by the results mentioned above.

First step is to pick $r_0 \in \mathbb{N}$ large enough and tile $\mathbb{Z}^2$ (or finite tori) with boxes of side lengths in $\{r_0, r_0 + 1\}$. This is possible in $O(\log^* n)$, or $O(\log^* 1/\varepsilon)$, rounds. The second step is to fix the randomness in such a way that:

- each vertex can compute its randomness in constant number of rounds independent of n,
- there is $R \in \mathbb{N}$ independent of n so that each vertex v on the boundary of the tiling needs to look to distance at most R to simulate $\mathcal{A}$ and output a solution,
- the solution given by $\mathcal{A}$ on the boundary can be extended to the interior of each tile.

Having this is clearly enough, as having a solution on the boundary allows to extend it inside each tile in constantly many rounds, once we know that such a solution exists.

References

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[HSW17] Alexander E. Holroyd, Oded Schramm, and David B. Wilson. Finitary coloring. *Ann. Probab.*, 45(5):2867–2898, 2017.