3 Deterministic vs Randomized complexities

**Theorem 0.1** (Linial [Lin92]). Let \( G \) be the class of all graphs of degree bounded by \( \Delta \) and \( \Pi \) be the proper vertex \((\Delta + 1)\)-coloring problem. Then we have
\[
\Pi \in \text{LOCAL}(O(\log^* n)).
\]

**Proof Sketch.** The proof is based on an iterated application of the following combinatorial result on cover free set systems.

**Claim 0.2** (Section 4.8.2 in [? ]). For every \( n > \Delta > 2 \) there is \( m \leq 4(\Delta + 1)^2 \log^2(n) \) and a family of subsets \( (S_i)_{i \in [n]} \) of \([m]\) such that
\[
S_{i_0} \not\subseteq \bigcup_{\ell \in [\Delta]} S_{i_\ell},
\]
whenever \( i_0 \neq i_\ell \) for every \( \ell \in [\Delta] \).

Start with a graph \( G \) on \( n \) vertices with unique identifiers from \( \text{poly}(n) \) and then use **Claim 0.2** iteratively to decrease the colors to \( \text{poly}(\Delta) \) in \( \log^* n \) rounds as follows. Suppose that we have coloring \( c_x \) with \( x \)-colors and \( x \) is larger than e.g. \( (11(\Delta + 1))^3 \), see [? , Section 4.8.3]. Pick \( m \) and \( (S_i)_{i \in [x]} \) as in **Claim 0.2**. In one communication round let \( c'(v) \) be the lexicographically minimal color in
\[
S_{c_x(v)} \setminus \bigcup_{(v,w) \in E(G)} S_{c_x(w)}.
\]
Then \( c' \) is a well defined \( m \)-coloring. It can be shown that as long as \( \log^{(i)} x \geq \log 36 + 2 \log(\Delta + 1) \), then we decrease the number of colors from
\[
(6(\Delta + 1)^2 \log^{(i-1)} x)^2 \text{ to } (6(\Delta + 1) \log^{(i)} x)^2
\]
in the \( i \)-th step, where \( i > 2 \).

Last step, as we treat \( \Delta \) as a constant, is to use the greedy algorithm. In general, this takes as many rounds as we have colors, in our case \( \text{poly}(\Delta) \).

Another general deterministic speed up is from \( o(\log \log^* n) \) to \( O(1) \), [NS95, CP17].

**Randomness.** Every deterministic local algorithm can be simulated as a randomized algorithm of the same complexity.

**Claim 0.3.** Let \( \Pi \in \text{LOCAL}(O(f(n))) \). Then \( \Pi \in \text{RLOCAL}(O(f(n))) \).

**Proof.** Let \( m \geq n \) and split \([0,1]\) into \( m \)-intervals \((I_j)_{j}\) of the same size. Let \( v \neq w \in V(G) \). Then the probability that their labels do not differ is \( 1/m \). By the union bound, we have that the probability of having a pair with a label in the same interval is at most \( n^2/m \). Consequently, taking \( m = n^3 \) yields the result.

Results in the opposite direction are called derandomization. It is known that this can be done in the regime \( o(\log \log n) \), [CKP16]. In particular cases, e.g., grids this can be improved to \( o(\log n) \).
**Proposition 0.4** (Corollary 1 in [? ]). Let \(G\) be closed under disjoint unions of finite graphs, or suppose that each \(G \in G_n\) can be found as an induced subgraph of some graph in \(G_m\) for every \(m > n\). Then

\[
\text{RLOCAL}(O(f(n))) = \text{LOCAL}(O(f(n)))
\]

for every \(f \in O(\log^* n)\).

**Proof Sketch.** Let \(A\) be a randomized local algorithm of complexity \((r_n)n \in O(\log^* n)\) that solves an LCL \(\Pi\) of locality \(t \in \mathbb{N}\). Fix \(n \in \mathbb{N}\). The idea is to find a (deterministic) function \(\varphi : [n] \rightarrow [0, 1]\) and \(m(n) > n\) such that running \(A\) for \(r m(n)\) many rounds on any \(G \in G_n\) labeled with images of unique identifiers under \(\varphi(-)\) produces \(\Pi\)-coloring. This will certainly produce a deterministic local algorithm of complexity \((r m(n))_n\).

We show that \(m(n) = 2n^2\) works as required. Note that in that case, if \(f \in O(\log^* n)\), then \(f(m(n)) \in O(f(n))\). Let \(G_n \in G_{m(n)}\) be a graph that contains as induced subgraph the graph \(H_n\) that is created by taking disjoint copies of all graphs from \(G_n\) labeled with unique identifiers. Note that \(|H_n| \ll m(n)\). Pick \(\varphi\) uniformly at random. By the definition the probability of the event that \(A_{m(n)}\) fails at a given vertex \(v \in V(G)\) for some \(G \in G_n\) (labeled with images of unique identifiers under \(\varphi\)) is at most \(\frac{1}{m(n)}\) by the definition. As \(|H_n| \ll m(n)\) we get by the union bound that there exists \(\varphi\) that does not fail at any vertex.

In general, randomness does help in the regime \(o(\log n)\). Picture. For example, deterministic complexity of the sinkless orientation problem on trees is \(\Omega(\log n)\), however, randomized complexity is \(O(\text{poly}(\log \log n))\). This is the most significant complexity class that is tightly connected with the distributed complexity of solutions to the Lovász Local Lemma (LLL). On a high level, given a randomized algorithm \(A\) that solves \(\Pi\) in \(o(\log n)\) rounds. We consider the bad events around vertices, when \(A\) fails. Then fixing \(k\) large enough we try to avoid bad events of \(A\) when applied to graphs of size \(k\). Note that uniformly the probability of bad event \(p\) is less than \(\frac{1}{k}\) and the degree \(d\) of bad event in the auxiliary graph of bad events is at most \(\Delta_o(\log k) \ll k\). That is the LLL condition \(p \cdot d \ll 1\) is satisfied. More on this later.

**References**


