1 Local problems

Our aim is to study various coloring problems on graphs. Let $\mathbf{G}$ be a class of some graphs (finite or infinite) of degree bounded by some fixed $\Delta < \infty$ possibly with some structure, e.g. vertex, edge labeling etc. Let $\mathbf{G}_\bullet$ be a class of rooted neighborhoods of graphs from $\mathbf{G}$. That is, if $H \in \mathbf{G}$, $v \in V(H)$ and $r \in \mathbb{N}$, then $\mathcal{B}_H(v, r) \in \mathbf{G}_\bullet$, where $\mathcal{B}_H(v, r)$ is the rooted neighborhood of v in H of diameter r (here we abuse the notation and instead of $(\mathcal{B}_H(v, r), v) \in \mathbf{G}_\bullet$ we write simply $\mathcal{B}(B, v) \in \mathbf{G}_\bullet$).

Examples of G:

- finite oriented cycles, infinite oriented line,
- finite tori of side lengths $O(\sqrt{n})$, infinite grid,
- finite trees of degree at most Δ, infinite Δ-regular tree,
- all (finite) graphs of degree bounded by Δ,
- all graphs with growth bounded by some $f : \mathbb{N} \to \mathbb{N}$,
- any of the above with vertices labeled by real numbers from $[0, 1]$, or with unique natural number from $[|V(G)|]$.

Sometimes we would like to add some labeling c to the structure of a given graph H. In that case, we write H_c to denote H together with the labeling c.

We start with a general definition of local problems. Note that this definition captures various notions of list colorings etc.

Definition 1.1 (LCLs with inputs, Naor and Stockmeyer [NS95]). *A locally checkable labeling problem (LCL) on* $\mathbf{G}$ *is a quadruple* $\Pi = (t, \Sigma_{in}, \Sigma_{out}, \mathcal{P})$, *where* $t \in \mathbb{N}$ *is the locality,* $\Sigma_{in}, \Sigma_{out}$ *are finite alphabets and* $\mathcal{P}$ *is a map from* $\Sigma_{in} \times \Sigma_{out}$ *vertex labeled rooted graphs from* $\mathbf{G}_\bullet$ *of diameter* t *to* $\{\text{true}, \text{false}\}$.

A vertex coloring (labeling) $c : G \to \Sigma_{in} \times \Sigma_{out}$ *of a graph* $G \in \mathbf{G}$ *is a* Π-coloring *if (formally)*

$$\mathcal{P}(c \restriction B_G(v, t)) = \text{true}$$

for every $v \in V(G)$.

To *solve* an LCL Π, we mean the following: given a graph $G \in \mathbf{G}$ with some input labeling $i : V(H) \to \Sigma_{in}$, we want to find an output labeling $o : V(H) \to \Sigma_{out}$ so that the pair (i, o) is a Π-coloring. Therefore whenever we talk about an LCL Π, we always assume that it is *non-trivial*, in the sense that every Σ_{in}-labeling of $H \in \mathbf{G}$ can be extended to a $\Sigma_{in} \times \Sigma_{out}$-coloring that is a Π-coloring.

Example 1.2. *The problem* 3-coloring-of-blocks *on oriented paths is defined as follows. There is binary input* $\Sigma_{in} = \{L, \emptyset\}$ *that should be interpreted as inducing consecutive blocks on the input path (cycle). The output set is* $\Sigma_{out} = \{\mathtt{R}, \mathtt{G}, \mathtt{B}\}$. *The problem asks to provide a 3-coloring of the given blocks. That is, the solution is checked as follows:* $\mathcal{P}$ *accepts* $((*, \emptyset) \times (\mathtt{A}, \mathtt{A}))$ *for* $\mathtt{A} \in \{\mathtt{R}, \mathtt{G}, \mathtt{B}\}$ *(the nodes in the same block agree on their color) and* $((*, L), (\mathtt{A}, \mathtt{B})$ *for* $\mathtt{A}, \mathtt{B} \in \{\mathtt{R}, \mathtt{G}, \mathtt{B}\}$ *and* $\mathtt{A} \neq \mathtt{B}$ *(neighboring blocks have different color).*

This is not solvable by any local algorithm as there might be just one block etc.

It might be interesting to only consider input labels that satisfy some global condition but all such results so far are very case specific. As an example consider the Circle squaring problem, here the underlying graph is a grid labeled with three colors, one corresponds to elements in the circle, one to elements in the square and the third to elements in the complement. The global constraint that we put on the labeling is that in any large box we can see the same fraction of circle and square labeled elements. The task is now to find a perfect matching in a power graph of the grid between square and circle labeled vertices.

References

[NS95] Moni Naor and Larry Stockmeyer. What can be computed locally? *SIAM Journal on Computing*, 24(6):1259–1277, 1995.