Traces for star products on symplectic manifolds

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Abstract

We give a direct elementary proof of the existence of traces for arbitrary star products on a symplectic manifold. We follow the approach we used in [9], solving first the local problem. A normalisation introduced by Karabegov [10] makes the local solutions unique and allows them to be pieced together to solve the global problem.

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1 Introduction

In a previous paper [9], we gave lowbrow proofs of some properties of differential star products on symplectic manifolds (in particular the classification of equivalence classes of such star products) using Čech cohomology methods and the existence of special local derivations which we called ν-Euler derivations in [9]. In this note, we present, in a similar spirit, properties of existence and uniqueness of traces for such star products. Our proof of the existence of a trace relies on a canonical way of normalization of the trace introduced by Karabegov [10], using local ν-Euler derivations.

Let $*$ be a star product (which we always assume here to be defined by bidifferential operators) on a symplectic manifold (M,ω). In the algebra of smooth functions on M, consider the ideal $C_0^\infty(M)$ of compactly supported functions. A *trace* is a $\mathbb{C}[\![\nu]\!]$-linear map $\tau\colon C_0^\infty(M)[\![\nu]\!] \to \mathbb{C}[\nu^{-1},\nu]\!]$ satisfying

$$\tau(u * v) = \tau(v * u).$$

The question of existence and uniqueness of such traces has been solved by the following result.

Theorem 4.1 (Fedosov [4, 5]; Nest–Tsygan [11]) *Any star product on a symplectic manifold (M,ω) has a trace which is unique up to multiplication by an element of $\mathbb{C}[\nu^{-1},\nu]\!]$. Every trace is given by a smooth density $\rho \in C^\infty(M)[\nu^{-1},\nu]\!]$:*

$$\tau(u) = \int_M u\rho \frac{\omega^n}{\nu^n n!}.$$

We shall give here an elementary proof of this theorem. The methods use intrinsically that we have a symplectic manifold. For Poisson manifolds Felder and Shoikhet have shown in [6] that the Kontsevich star product also has a trace.

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2 Traces

Let (M,ω) be a connected symplectic manifold, N the algebra of smooth functions and N_c the ideal in N of compactly supported functions. Obviously, $N_c[\![\nu]\!]$ is an ideal in $N[\![\nu]\!]$ and any differential star product or equivalence on $N[\![\nu]\!]$ is determined on $N_c[\![\nu]\!]$.

Definition 2.1 Let $*$ be a star product on (M,ω) then a *trace* is a $\mathbb{C}[\![\nu]\!]$-linear map $\tau\colon N_c[\![\nu]\!] \to \mathbb{C}[\nu^{-1},\nu]\!]$ satisfying

$$\tau(u * v) = \tau(v * u).$$

Remark 2.2 Since any $\mathbb{C}[\nu^{-1}, \nu]\!]$-multiple of a trace is a trace, it is not necessary to work with Laurent series, but we do so for two reasons. Firstly, in the formula for the trace of a pseudo-differential operator a factor of ν^{-n} occurs, and secondly, the presence of such a factor simplifies equation (3) below.

If τ is a non-trivial trace, we take u in N_c and can then expand

$$\tau(u) = \nu^r \sum_{s \geq 0} \nu^s \tau_s(u)$$

where each $\tau_s \colon N_c \to \mathbb{R}$ is a linear map and we assume $\tau_0 \neq 0$. The condition to be a trace takes the form

$$\tau_k(\{u, v\}) + \tau_{k-1}(C_2^-(u, v)) + \ldots + \tau_0(C_{k+1}^-(u, v)) = 0 \tag{1}$$

for $k = 0, 1, 2, \ldots$, where C_r^- denotes the antisymmetric part of C_r.

Remark 2.3 In [3] the notion of a *closed star product* was introduced and related to cyclic cohomology of the algebra of functions. The existence of a closed star product was proved in [12]; see also [1, 13]. A star product is closed if

$$\int_M C_r^-(u, v)\omega^n = 0, \qquad \forall u, v \in N_c \tag{2}$$

for all $1 \leq r \leq n$ where $2n$ is the dimension of M. If (2) holds for all $r \geq 1$ then $u * v$ is said to be *strongly closed*. Thus to be strongly closed is the same as requiring that

$$\tau(u) = \int_M u\omega^n$$

be a trace on $N_c[\![\nu]\!]$.

For $k = 0$, equation (1) reduces to the condition $\tau_0(\{u, v\}) = 0$ for all u, v. In [2] it is shown by elementary means that this implies that τ_0 is a multiple of the integral $\int_M u\omega^n$ when M is connected. We give a proof here for completeness.

Lemma 2.4 (Gelfand & Shilov [7]) *If u is a compactly supported smooth function on $\mathbb{R}^N$ with $\int_{\mathbb{R}^N} u\, d^N x = 0$ then u is a sum of derivatives of compactly supported smooth functions.*

Proof For $N = 1$ we simply set $v(x) = \int_{-\infty}^{x} u(t)dt$ and observe that v obviously vanishes when x is below the support of u, and is zero again when x is large as a consequence of $\int_{-\infty}^{\infty} u(t)dt = 0$. Thus v is compactly supported and $u(x) = \frac{dv}{dx}$. Moreover, if u depends smoothly on parameters, so will v and if u is compactly supported in the parameters, so is v.

Now proceed by induction on N. $w(x_1, \ldots, x_{N-1}) = \int_{-\infty}^{\infty} u(x_1, \ldots, x_{N-1}, t)\, dt$ clearly is compactly supported in $\mathbb{R}^{N-1}$ and has vanishing integral, so by the inductive assumption $w(x_1, \ldots, x_{N-1}) = \sum_{i=1}^{N-1} \frac{\partial w_i(x_1, \ldots, x_{N-1})}{\partial x_i}$ for some compactly supported functions w_i on $\mathbb{R}^{N-1}$. Take a compactly supported bump function $r(t)$ on $\mathbb{R}$ with $\int_{\mathbb{R}} r(t)\, dt = 1$ and consider $u(x_1, \ldots, x_N) - w(x_1, \ldots, x_{N-1})r(x_N)$ which is compactly supported in all its variables. Integrating in x_N we see that the integral over $\mathbb{R}$ vanishes, and so $u(x_1, \ldots, x_N) - w(x_1, \ldots, x_{N-1})r(x_N) = \frac{\partial v}{\partial x_N}(x_1, \ldots, x_N)$. Thus $u = \frac{\partial v}{\partial x_N}(x_1, \ldots, x_N) + \sum_{i=1}^{N-1} \frac{\partial w_i(x_1, \ldots, x_{N-1})r(x_N)}{\partial x_i}$ which completes the inductive step. □

Lemma 2.5 (Bordemann, Römer, Waldmann [2]) *Let (M, ω) be a connected symplectic manifold. If $\sigma \colon N_c \to \mathbb{R}$ is a linear map with $\sigma(\{u, v\}) = 0$ for all $u, v \in N_c$ then $\sigma(u) = c \int_M u\omega^n$ for some constant c.*

PROOF Fix $u \in N_c$ and cover M by Darboux charts U_α such that only finitely many U_α intersect the support of u. Take a partition of unity φ_α subordinate to U_α then only a finite number of $u\varphi_\alpha$ are non-zero. Thus $\sigma(u) = \sum_\alpha \sigma(u\varphi_\alpha)$.

$u\varphi_\alpha - \frac{\int_M u\varphi_\alpha\omega^n}{\int_M \varphi_\alpha\omega^n}\varphi_\alpha$ has vanishing integral on U_α which can be viewed as an open set in some $\mathbb{R}^{2n}$. Thus by the previous Lemma there are functions v_i, w_i with compact support such that $u\varphi_\alpha - \frac{\int_M u\varphi_\alpha\omega^n}{\int_M \varphi_\alpha\omega^n}\varphi_\alpha = \sum_i \frac{\partial v_i}{\partial p_i} + \frac{\partial w_i}{\partial q_i}$ for the Darboux coordinates p_i and q_i. But $\frac{\partial v_i}{\partial p_i} = \{v_i, q_i\}$, and if we choose a function s_i of compact support which is identically 1 on the support of v_i then $\frac{\partial v_i}{\partial p_i} = \{v_i, s_i q_i\}$ so we see that $\sigma\left(\frac{\partial v_i}{\partial p_i}\right) = 0$ and similarly $\sigma\left(\frac{\partial w_i}{\partial q_i}\right) = 0$. Thus $\sigma(u\varphi_\alpha) = \frac{\int_M u\varphi_\alpha\omega^n}{\int_M \varphi_\alpha\omega^n}\sigma(\varphi_\alpha)$. Hence

$$\begin{aligned} \sigma(u) &= \int_M u \sum_\alpha \frac{\varphi_\alpha}{\int_M \varphi_\alpha\omega^n}\sigma(\varphi_\alpha)\omega^n \\ &= \int_M u\rho\omega^n. \end{aligned}$$

Since $\sigma(\{u, v\}) = 0$, $\int_M \{u, v\}\rho\omega^n = 0$ and hence $\int_M \{\rho, u\}v\omega^n = 0$ for all v. Thus $\{\rho, u\} = 0$ for all u and hence $\rho = c$, a constant. □

Thus any trace has the form

$$\tau(u) = a\nu^r \left(\int_M u\omega^n + \sum_{k\geq 1} \nu^k \tau_k(u)\right)$$

where $a \neq 0$. We can divide by a and multiply by ν^{-n-r} to bring τ into the form

$$\int_M u\frac{\omega^n}{\nu^n} + \nu^{-n}\sum_{k\geq 1}\nu^k\tau_k(u).$$

Any trace in this form will be said to be *standard* c.f. [8].

As observed in [2], this is enough to show that any two traces τ and τ' for the same star product are proportional. For, if τ is standard, the leading term of τ' is a multiple $c\nu^r$ of the integral, hence is equal to the leading term of τ multiplied by $c\nu^{r+n}$. But then $\tau' - c\nu^{r+n}\tau$ is a trace which vanishes to at least order $r+1$ in ν. This argument can be repeated indefinitely to show that $\tau' = c\nu^{r+n}(1+\sum_k \nu^k c_k)\tau$. Remark that in particular if τ and τ' are standard then $\tau' = (1+\sum_k \nu^k c_k)\tau$. This proves

Theorem 2.6 (Nest & Tsygan [11]) *On a connected symplectic manifold (M,ω) any two traces are proportional by an element of $\mathbb{C}[\nu^{-1},\nu]\!]$.*

3 The local case

In the case of $\mathbb{R}^{2n}$ with its standard constant 2-form Ω, then

$$\tau_M(u) = \int_{\mathbb{R}^{2n}} u\,\frac{\Omega^n}{\nu^n n!}$$

is a trace on compactly supported functions for the Moyal star product $*_M$. The Moyal star product and this trace have an important homogeneity property [10]. If we take a conformal vector field ξ on $\mathbb{R}^{2n}$, so $\mathcal{L}_\xi\Omega = \Omega$ then $D_M = \xi + \nu\frac{\partial}{\partial\nu}$ is a derivation of $*_M$ and τ_M satisfies

$$\tau_M(D_M u) = \nu\frac{\partial}{\partial\nu}\tau_M(u). \tag{3}$$

If $*$ is any star product defined on an open ball U in $\mathbb{R}^{2n}$ then it is equivalent to the restriction of the Moyal star product by a map $T = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k$ with

$$T(u*v) = T(u) *_M T(v)$$

and then we see that

$$\tau(u) = \int_U T(u)\frac{\Omega^n}{\nu^n}$$

is a trace for $*$. Each T_k is a differential operator, so it has a formal adjoint T_k' so that, if we put $T' = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k'$, then

$$\tau(u) = \int_U uT'(1)\frac{\Omega^n}{\nu^n}$$

for $u \in C_c^\infty(U)$. If we put $\rho = T'(1) \in C^\infty(U)[\nu^{-1},\nu]\!]$ then

$$\tau(u) = \int_U u\rho\frac{\Omega^n}{\nu^n}.$$

If τ is standard then $\rho = 1 + \sum_{k\geq 1} \nu^k \rho_k$.

Further $D = T^{-1} \circ D_M \circ T$ will be a derivation of $*$ and satisfies the transform of equation (3):

$$\tau(Du) = \nu \frac{\partial}{\partial \nu} \tau(u).$$

D has the form $\xi + \nu \frac{\partial}{\partial \nu} + D'$. Local derivations of this form we give a special name

Definition 3.1 Let (M, ω) be a symplectic manifold. Say that a derivation D on an open set U of $N[\![\nu]\!], *$ is **ν-Euler** if it has the form

$$D = \nu \frac{\partial}{\partial \nu} + X + D' \tag{4}$$

where X is conformally symplectic ($\mathcal{L}_X \omega = \omega$) and $D' = \sum_{r\geq 1} \nu^r D'_r$ with the D'_r differential operators on U.

Note that conformally symplectic vector fields only exist locally in general, so we also cannot ask for global ν-Euler derivations. Two local ν-Euler derivations defined on the same open set will differ by a ν-linear derivation, and so the difference is $\nu^{-1} \times$ inner. This means that $\tau \circ D$ will be independent of D and thus is globally defined as an $\mathbb{R}$-linear functional, even if D is not.

Definition 3.2 A standard trace τ is *normalised* if it satisfies the analogue of the Moyal homogeneity condition:

$$\tau(Du) = \nu \frac{\partial}{\partial \nu} \tau(u)$$

on any open set where there are ν-Euler derivations and for any such local ν-Euler derivation D.

This condition was introduced by Karabegov [10].

It is clear that the pull-back of the Moyal trace by an equivalence with the Moyal star product is normalised, so a normalised trace always exists for any star product on an open ball in $\mathbb{R}^{2n}$.

Proposition 3.3 (Karabegov [10]) *If τ and τ' are normalised traces for the same star product on an open ball U in $\mathbb{R}^{2n}$ then $\tau = \tau'$.*

Proof Neither trace can be zero, and are proportional so $\tau' = (1 + c\nu^r + \ldots)\tau$. If $\tau' \neq \tau$ then there is a first $r > 0$ where $c \neq 0$. Then we substitute in the normalisation condition to give

$$\begin{aligned}
(1 + c\nu^r + \ldots)\tau(Du) &= \tau'(Du) \\
&= \nu \frac{\partial}{\partial \nu} \tau'(u)
\end{aligned}$$

$$\begin{aligned} &= \nu\frac{\partial}{\partial\nu}((1+c\nu^r+\ldots)\tau(u)) \\ &= (rc\nu^r+\ldots)\tau(u)+(1+c\nu^r+\ldots)\tau(Du) \end{aligned}$$

which implies that $c = 0$. This contradiction shows that $\tau' = \tau$. $\square$

4 The global case

Let (M, ω) be a connected symplectic manifold. Then we can cover M by Darboux charts U which are diffeomorphic to open balls in $\mathbb{R}^{2n}$ and such that all non-empty intersections are also diffeomorphic to open balls. Let $*$ be a star product on M and then the restriction of $*$ to U has a normalised trace with density $\rho_U \in C^\infty(U)[\![\nu]\!]$. If we have two open sets U, V which overlap, then on the intersection both ρ_U and ρ_V will determine normalised traces on an open ball in $\mathbb{R}^{2n}$. Since there is only one such trace, $\rho_U = \rho_V$ on $U \cap V$. It follows that there is a globally defined function ρ on M such that $\rho_U = \rho|_U$. Set

$$\tau(u) = \int_M u\rho\frac{\omega^n}{\nu^n}$$

for $u \in C_c^\infty(M)$.

Given u, v in $C_c^\infty(M)$ we can find a finite partition of unity φ_i on $\operatorname{supp} u \cup \operatorname{supp} v$ with supports in the open sets above. Then $u = \sum_i \varphi_i u$ and $v = \sum_j \varphi_j v$ so $u * v = \sum_{i,j}(\varphi_i u) * (\varphi_j v)$ so $\tau(u * v) = \sum_{i,j}\tau((\varphi_i u) * (\varphi_j v)) = \sum_{i,j}\tau((\varphi_j v) * (\varphi_i u)) = \tau(v * u)$ so τ is a trace. A similar partition of unity argument shows that τ is normalised.

Combining this with Theorem 2.6 and the fact that the normalised trace we just constructed has a smooth density we obtain:

Theorem 4.1 (Fedosov, Nest–Tsygan) *On a connected symplectic manifold (M, ω) any differential star product has a unique normalised trace. Any trace is multiple of this and is given by a smooth density.*

Remark 4.2 In [11] a proof that any trace has a smooth density is given using cyclic cohomology.

Corollary 4.3 *Any trace is invariant under all $\mathbb{C}[\![\nu]\!]$-linear automorphisms of the star product.*

PROOF A smooth trace is a multiple (in $\mathbb{C}[\nu^{-1}, \nu]\!]$) of a normalised trace. The transform of a ν-Euler derivation by a $\mathbb{C}[\![\nu]\!]$-linear automorphism is again a ν-Euler derivation, thus the transform of a normalised trace by a $\mathbb{C}[\![\nu]\!]$-linear automorphism is again a normalised trace, and so is equal to the original normalised trace. $\square$

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